

A TRANSFERENCE RESULT OF THE L^p CONTINUITY OF THE JACOBI LITTLEWOOD-PALEY g -FUNCTION TO THE GAUSSIAN AND LAGUERRE LITTLEWOOD-PALEY g -FUNCTION.

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ABSTRACT. In this paper we develop a transference method to obtain the L^p -continuity of the Gaussian-Littlewood-Paley g function and the L^p -continuity of the Laguerre-Littlewood-Paley g function from the L^p -continuity of the Jacobi-Littlewood-Paley g function, in dimension one, using the well known asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials.

1. PRELIMINARIES

It is well known, in the theory of classical orthogonal polynomials, the asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials. Using those asymptotic relations we develop a transference method to obtain L^p -continuity for the Gaussian-Littlewood-Paley g function and the L -continuity of the Laguerre-Littlewood-Paley g function from the L^p -continuity of the Jacobi-Littlewood-Paley g function, in dimension one. We are going to use the normalizations given in G. Szegő's book [9], for all classical polynomials.

- *Jacobi polynomials:* For $\alpha, \beta > -1$, the Jacobi polynomials $\{P_n^{(\alpha, \beta)}\}_{n \in \mathbb{N}}$ are defined as the orthogonal polynomials associated with the Jacobi measure $\mu_{\alpha, \beta}$ (or beta measure) in $(-1, 1)$, defined as

$$\mu_{\alpha, \beta}(dx) = \omega_{\alpha, \beta}(x)dx = \eta_{\alpha, \beta} \chi_{(-1, 1)}(x) (1-x)^\alpha (1+x)^\beta dx,$$

$$\text{where } \eta_{\alpha, \beta} = \frac{1}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1)}.$$

The function $\omega_{\alpha, \beta}$ is called the (normalized) *Jacobi weight*.

The Jacobi polynomials can be obtained from the polynomial canonical basis $\{1, x, x^2, \dots, x^n, \dots\}$ using the Gram-Schmidt orthogonalization process with respect to the inner product in $L^2(\mu_{\alpha, \beta})$. Thus we have the *orthogonality property* of Jacobi polynomials with respect to $\mu_{\alpha, \beta}$,

$$(1.1) \quad \int_{-\infty}^{\infty} P_n^{(\alpha, \beta)}(y) P_m^{(\alpha, \beta)}(y) \mu_{\alpha, \beta}(dy) = \eta_{\alpha, \beta} h_n^{(\alpha, \beta)} \delta_{n, m} = \hat{h}_n^{(\alpha, \beta)} \delta_{n, m},$$

$n, m = 0, 1, 2, \dots$, where

$$(1.2) \quad h_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{(2n + \alpha + \beta + 1)} \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)},$$

and

$$\begin{aligned} \hat{h}_n^{(\alpha, \beta)} &= \frac{1}{(2n + \alpha + \beta + 1)} \frac{\Gamma(\alpha + \beta + 2) \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(\alpha + 1) \Gamma(\beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)} \\ &= \|P_n^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2. \end{aligned}$$

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On the other hand, the Jacobi polynomial of parameter (α, β) of degree n , $P_n^{(\alpha, \beta)}$, is a polynomial solution of the *Jacobi differential equation*, with parameters α, β, n ,

$$(1.3) \quad (1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0,$$

i.e. $P_n^{(\alpha, \beta)}$ is an eigenfunction of the (one-dimensional) second order diffusion operator

$$(1.4) \quad \mathcal{L}^{\alpha, \beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta - \alpha - (\alpha + \beta + 2)x)\frac{d}{dx},$$

associated with the eigenvalue $\lambda_n^{\alpha, \beta} = n(n + \alpha + \beta + 1)$. $\mathcal{L}^{\alpha, \beta}$ is called the *Jacobi differential operator*. Observe that if we choose $\delta_{\alpha, \beta} = \sqrt{1-x^2}\frac{d}{dx}$, and consider its formal $L^2(\mu_{\alpha, \beta})$ -adjoint,

$$\delta_{\alpha, \beta}^* = -\sqrt{1-x^2}\frac{d}{dx} + [(\alpha + \frac{1}{2})\sqrt{\frac{1+x}{1-x}} - (\beta + \frac{1}{2})\sqrt{\frac{1-x}{1+x}}]I,$$

then $\mathcal{L}^{\alpha, \beta} = \delta_{\alpha, \beta}^* \delta_{\alpha, \beta}$. The differential operator $\delta_{\alpha, \beta}$ is considered the “natural” notion of derivative in the Jacobi case.

The operator semigroup associated to the Jacobi polynomials is defined for positive or bounded measurable Borel functions of $(-1, 1)$, as

$$(1.5) \quad T_t^{\alpha, \beta} f(x) = \int_{-1}^1 p^{\alpha, \beta}(t, x, y) f(y) \mu_{\alpha, \beta}(dy),$$

where

$$p^{\alpha, \beta}(t, x, y) = \sum_k \frac{e^{-k(k+\alpha+\beta+1)t}}{\hat{h}_k^{(\alpha, \beta)}} P_k^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(y).$$

There is not a simple explicit representation of $p^{\alpha, \beta}(t, x, y)$ since the eigenvalues λ_n are not linearly distributed; there is one obtained by G. Gasper [3] which is analog of Bailey’s F_4 representation of the kernel of Abel summability for Jacobi series, also called the Jacobi-Poisson integral, see [1]. From that form, taking $x = -y = 1$, it can be proved that $p^{\alpha, \beta}(t, x, y)$ is a positive kernel.

$\{T_t^{\alpha, \beta}\}$ is called the *Jacobi semigroup* and can be proved that is a Markov semigroup, for details see [10]. The Jacobi-Poisson semigroup $\{P_t^{\alpha, \beta}\}$ can be defined, using Bochner’s subordination formula,

$$e^{-\lambda^{1/2}t} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\lambda^2}{4u}} du.$$

as the subordinated semigroup of the Jacobi semigroup,

$$P_t^{\alpha, \beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^{\alpha, \beta} f(x) du.$$

For a function $f \in L^2([-1, 1], \mu_{(\alpha, \beta)})$ let us consider its Fourier- Jacobi expansion

$$(1.6) \quad f = \sum_{k=0}^\infty \frac{\langle f, P_k^{(\alpha, \beta)} \rangle}{\hat{h}_k^{(\alpha, \beta)}} P_k^{(\alpha, \beta)},$$

where

$$\langle f, P_k^{(\alpha, \beta)} \rangle = \int_{-1}^1 f(y) P_k^{(\alpha, \beta)}(y) \mu_{\alpha, \beta}(dy).$$

Then, the action of T_t and P_t can be expressed as

$$T_t^{\alpha,\beta} f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} e^{-\lambda_k t} P_k^{(\alpha,\beta)},$$

and

$$P_t^{\alpha,\beta} f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} e^{-\sqrt{\lambda_k} t} P_k^{(\alpha,\beta)}.$$

Following the classical case, the Jacobi-Littlewood-Paley g function can be define as

$$(1.7) \quad g^{(\alpha,\beta)} f(x) = \int_0^\infty t |\nabla_{(\alpha,\beta)} P_t^{(\alpha,\beta)} f(x)|^2 dt$$

where $\nabla_{(\alpha,\beta)} = (\frac{\partial}{\partial t}, \delta_{\alpha,\beta}) = (\frac{\partial}{\partial t}, \sqrt{1-x^2} \frac{\partial}{\partial x})$

The L^p -continuity of the Jacobi-Littlewood-Paley g -function $g(\alpha; \beta)$, was proved by A. Nowak and P. Sjögren in [8].

Theorem 1.1. *Assume that $1 < p < \infty$ and $\alpha, \beta \in [-1/2, \infty)^d$. There exists a constant c_p such that*

$$(1.8) \quad \|g^{(\alpha,\beta)} f\|_{p,(\alpha,\beta)} \leq c_p \|f\|_{p,(\alpha,\beta)}.$$

- *Hermite polynomials:* The *Hermite polynomials* $\{H_n\}_n$, are defined as the orthogonal polynomials associated with the Gaussian measure in \mathbb{R} , $\gamma(dx) = \frac{e^{-x^2}}{\sqrt{\pi}} dx$, i.e.

$$(1.9) \quad \int_{-\infty}^{\infty} H_n(y) H_m(y) \gamma(dy) = 2^n n! \delta_{n,m},$$

$n, m = 0, 1, 2, \dots$, with the *normalization*

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}.$$

We have

$$(1.10) \quad H'_n(x) = 2n H_{n-1}(x),$$

$$(1.11) \quad H''_n(x) - 2x H'_n(x) + 2n H_n(x) = 0,$$

thus H_n is an eigenfunction of the one dimensional *Ornstein-Uhlenbeck operator* (or harmonic oscillator operator),

$$(1.12) \quad L = -\frac{1}{2} \frac{d^2}{dx^2} + x \frac{d}{dx},$$

associated with the eigenvalue $\lambda_n = n$. Observe that if we choose $\delta_\gamma = \frac{1}{\sqrt{2}} \frac{d}{dx}$, and consider its formal $L^2(\gamma)$ -adjoint,

$$\delta_\gamma^* = -\frac{1}{\sqrt{2}} \frac{d}{dx} + \sqrt{2} x I$$

then $L = \delta_\gamma^* \delta_\gamma$. The differential operator δ_γ is considered the “natural” notion of derivative in the Hermite case.

The Gaussian-Littlewood-Paley g function can be defined as

$$(1.13) \quad g^\gamma f(x) = \int_0^\infty t |\nabla_\gamma P_t^\gamma f(x)|^2 dt$$

where $\nabla_\gamma = (\frac{\partial}{\partial t}, \delta_\gamma) = (\frac{\partial}{\partial t}, \frac{1}{\sqrt{2}} \frac{\partial}{\partial x})$

The L^p continuity of the Gaussian-Littlewood-Paley g function was proved by C. Gutiérrez in [5],

Theorem 1.2. *Assume that $1 < p < \infty$. There exists a constant c_p such that*

$$(1.14) \quad \|g^\gamma f\|_{p,\gamma} \leq c_p \|f\|_{p,\gamma}.$$

- *Laguerre polynomials:* For $\alpha > -1$, the *Laguerre polynomials* $\{L_k^\alpha\}$ are defined as the orthogonal polynomials associated with the Gamma measure on $(0, \infty)$, $\mu_\alpha(dx) = \chi_{(0,\infty)}(x) \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} dx$, i.e.

$$(1.15) \quad \int_0^\infty L_n^\alpha(y) L_m^\alpha(y) \mu_\alpha(dy) = \binom{n+\alpha}{n} \delta_{n,m} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} \delta_{n,m},$$

$n, m = 0, 1, 2, \dots$ We have

$$(1.16) \quad (L_k^\alpha(x))' = -L_{k-1}^{\alpha+1}(x).$$

$$(1.17) \quad x(L_k^\alpha(x))'' + (\alpha+1-x)(L_k^\alpha(x))' + kL_k^\alpha(x) = 0.$$

thus L_k^α is an eigenfunction of the (one-dimensional) *Laguerre differential operator*

$$(1.18) \quad \mathcal{L}^\alpha = -x \frac{d^2}{dx^2} - (\alpha+1-x) \frac{d}{dx},$$

associated with the eigenvalue $\lambda_k = k$. Observe that if we choose $\delta_\alpha = \sqrt{x} \frac{d}{dx}$, and consider its formal $L^2(\alpha)$ -adjoint,

$$\delta_\alpha^* = -\sqrt{x} \frac{d}{dx} + \left[\frac{\alpha+1/2}{\sqrt{x}} + \sqrt{x} \right] I$$

then $\mathcal{L}^\alpha = \delta_\alpha^* \delta_\alpha$. The differential operator δ_α is considered the “natural” notion of derivative in the Laguerre case.

The Laguerre-Littlewood-Paley g function can be defined as

$$(1.19) \quad g^\alpha f(x) = \int_0^\infty t |\nabla_\alpha P_t^\alpha f(x)|^2 dt$$

where $\nabla_\alpha = (\frac{\partial}{\partial t}, \delta_\alpha) = (\frac{\partial}{\partial t}, \sqrt{x} \frac{\partial}{\partial x})$.

The L^p continuity of the Laguerre-Littlewood-Paley g function was proved by A. Nowak in [7].

Theorem 1.3. *Assume that $1 < p < \infty$ and $\alpha \in [1/2, \infty)^d$. There exists a constant c_p such that*

$$(1.20) \quad \|g^\alpha f\|_{p,\alpha} \leq c_p \|f\|_{p,\alpha}.$$

- Finally, let us consider the asymptotic relations between Jacobi polynomials and other classical orthogonal polynomials (see [9], (5.3.4) and (5.6.3)),
i) For Hermite polynomials,

$$(1.21) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-n/2} C_n^\lambda(x/\sqrt{\lambda}) = \frac{H_n(x)}{n!},$$

- ii) For Laguerre polynomials,

$$(1.22) \quad \lim_{\beta \rightarrow \infty} P_n^{(\alpha,\beta)}(1-2x/\beta) = L_n^\alpha(x).$$

Both relations holds uniformly in every closed interval of \mathbb{R} .

Actually these relations are expression of deeper relations between the measures and operators involved. As a consequence of those relations we have the following technical results, that were proved in [6], and are needed to prove Theorem 2.1.

Proposition 1.1. (*norm relations*)

i) Let $f \in L^2(\mathbb{R}, \gamma)$ and define $f_\lambda(x) = f(\sqrt{\lambda}x)\chi_{[-1,1]}(x)$, then $f_\lambda \in L^2([-1,1], \mu_\lambda)$ and

$$(1.23) \quad \lim_{\lambda \rightarrow \infty} \|f_\lambda\|_{2,\lambda} = \|f\|_{2,\gamma}$$

ii) Let $f \in L^2(\mathbb{R}, \mu_\alpha)$ and define $f_\beta(x) = f\left(\frac{\beta}{2}(1-x)\right)\chi_{[-1,1]}(x)$, then $f_\beta \in L^2([-1,1], \mu_{(\alpha,\beta)})$ and

$$(1.24) \quad \lim_{\beta \rightarrow \infty} \|f_\beta\|_{2,(\alpha,\beta)} = \|f\|_{2,\alpha}$$

Proposition 1.2. (*inner product relations*) With the same notation as in Lemma 1.1,

i) Let $f \in L^2(\mathbb{R}, \gamma)$, then

$$(1.25) \quad \lim_{\lambda \rightarrow \infty} \langle f_\lambda, \lambda^{-k/2} C_k^\lambda \rangle = \langle f, \frac{H_k}{k!} \rangle.$$

ii) Let $f \in L^2(\mathbb{R}, \mu_\alpha)$, then

$$(1.26) \quad \lim_{\beta \rightarrow \infty} \langle f_\beta, P_k^{(\alpha,\beta)} \rangle = \langle f, L_k^\alpha \rangle.$$

The results of this paper follows the same scheme of the proof given in [6], where this transference method was used to obtain the the L^p boundedness for the Riesz transform in the Hermite and Laguerre case from the L^p boundedness for the Riesz transform in the Jacobi case. Unfortunately due to the non-linearity of the Littlewood-Paley g -function the computations for the case $p \neq 2$ are more involved.

2. MAIN RESULTS

We want to obtain the L^p -continuity for the Gaussian-Littlewood-Paley g and the L^p -continuity for the Laguerre-Littlewood-Paley g from the L^p -continuity of the Jacobi-Littlewood-Paley g , using a transference method based on the asymptotic relations between Jacobi polynomials and Hermite and Laguerre polynomials. We will start considering the case $p = 2$; more precisely we want to prove

Theorem 2.1. *The $L^2(\mu_{\alpha,\beta})$ boundedness for the Jacobi-Littlewood-Paley g*

$$(2.1) \quad \|g^{(\alpha,\beta)} f\|_{2,(\alpha,\beta)} \leq C_2 \|f\|_{2,(\alpha,\beta)}$$

implies

i) *the $L^2(\gamma)$ boundedness for the Gaussian-Littlewood-Paley g*

$$(2.2) \quad \|g^\gamma f\|_{2,\gamma} \leq C_2 \|f\|_{2,\gamma}.$$

ii) *the $L^2(\mu_\alpha)$ boundedness for the Laguerre-Littlewood-Paley g*

$$(2.3) \quad \|g^\alpha f\|_{2,\alpha} \leq C_2 \|f\|_{2,\alpha}.$$

Proof

i) Let $f \in L^2(\mathbb{R}, \gamma)$ and define $f_\lambda(x) = f(\sqrt{\lambda}x)$; then

$$\begin{aligned} \|g^{(\alpha,\beta)} f_\lambda\|_{2,(\alpha,\beta)}^2 &= \int_{-1}^1 \left(\int_0^\infty t |\nabla_{\alpha,\beta} P_t^{(\alpha,\beta)} f_\lambda(x)|^2 dt \right) d\mu_{\alpha,\beta}(x) \\ &= \int_0^\infty t \left(\int_{-1}^1 \left(\left(\frac{\partial}{\partial t} P_t^{(\alpha,\beta)} f_\lambda(x) \right)^2 + (1-x^2) \left(\frac{\partial}{\partial x} P_t^{(\alpha,\beta)} f_\lambda(x) \right)^2 \right) d\mu_{\alpha,\beta}(x) \right) dt \end{aligned}$$

Let us study the first term, by Parseval's identity,

$$\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\alpha,\beta)} f_\lambda(x) \right)^2 d\mu_{\alpha,\beta}(x) = \sum_{k=1}^{\infty} \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}},$$

Therefore, interchanging the integral with the series, using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\alpha, \beta)} f_\lambda(x) \right)^2 d\mu_{\alpha, \beta}(x) \right) dt &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}} \right) dt \\ &= \frac{1}{4} \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \right\rangle \right|^2 = \frac{1}{4} \|f_\lambda\|_{2, (\alpha, \beta)}^2 \end{aligned}$$

In particular, taking $\alpha = \beta = \lambda - 1/2$, we obtain

$$\int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt = \frac{1}{4} \int_{-1}^1 |f_\lambda(x)|^2 d\mu_\lambda(x) = \frac{1}{4} \|f_\lambda\|_{2, \lambda}^2$$

Thus, we get using Proposition 1.1

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt &= \lim_{\lambda \rightarrow \infty} \frac{1}{4} \|f_\lambda\|_{2, \lambda}^2 = \frac{1}{4} \|f\|_{2, \gamma}^2 \\ &= \frac{1}{4} \int_{-\infty}^\infty |f(x)|^2 d\gamma(x) = \frac{1}{4} \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2, \gamma}} \right\rangle \right|^2 = \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2, \gamma}} \right\rangle \right|^2 k \int_0^\infty t e^{-2tk^{1/2}} dt \\ &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2, \gamma}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt = \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right) d\gamma(x) \right) dt \end{aligned}$$

Hence,

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt = \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt.$$

Now, let us consider the second term. First of all, observe that

$$\begin{aligned} \left\| \sqrt{1-x^2} P_{k-1}^{(\alpha+1, \beta+1)} \right\|_{2, (\alpha, \beta)}^2 &= \frac{4(\alpha+1)(\beta+1)}{2^{\alpha+\beta+3}(\alpha+\beta+3)(\alpha+\beta+2)B(\alpha+2, \beta+2)} \\ &\quad \times \int_{-1}^1 (1-x)^{\alpha+1}(1+x)^{\beta+1} \left[P_{k-1}^{(\alpha+1, \beta+1)}(x) \right]^2 dx \\ &= \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)} \left\| P_{k-1}^{(\alpha+1, \beta+1)} \right\|_{2, (\alpha+1, \beta+1)}^2 = \frac{4k}{(k+\alpha+\beta+1)} \left\| P_k^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}^2. \end{aligned}$$

Then, using Parseval's identity, we get

$$\begin{aligned} \int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha, \beta)} f_\lambda(x) \right)^2 d\mu_{\alpha, \beta}(x) &= \left\| \sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha, \beta)} f_\lambda \right\|_{2, (\alpha, \beta)}^2 \\ &= \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \right\rangle \right|^2 e^{-2t\lambda_k^{1/2}} \frac{(k+\alpha+\beta+1)^2}{4} \left\| \sqrt{1-x^2} P_{k-1}^{(\alpha+1, \beta+1)} \right\|_{2, (\alpha, \beta)}^2 \\ &= \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \right\rangle \right|^2 e^{-2t\lambda_k^{1/2}} \lambda_k \left\| P_k^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}^2 = \sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}}. \end{aligned}$$

Now as before, using Lebesgue's dominated convergence theorem, interchanging the integral with the series, we get

$$\begin{aligned} & \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha,\beta)} f_\lambda(x) \right)^2 d\mu_{\alpha,\beta}(x) \right) dt \\ &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f_\lambda, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}} \right\rangle \right|^2 \lambda_k e^{-2t\lambda_k^{1/2}} \right) dt = \frac{1}{4} \int_{-1}^1 |f_\lambda(x)|^2 d\mu_{\alpha,\beta}(x). \end{aligned}$$

Then, for the Gegenbauer case, $\alpha = \beta = \lambda - 1/2$ we get

$$\int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\lambda-1/2,\lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt = \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2.$$

On the other hand, again using Parseval's identity, we have

$$\begin{aligned} & \int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) = \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 2k^2 e^{-2tk^{1/2}} \|H_{k-1}\|_{2,\gamma}^2 \\ &= \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 2k^2 e^{-2tk^{1/2}} \frac{1}{2k} \|H_k\|_{2,\gamma}^2 = \sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 k e^{-2tk^{1/2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{H_k}{\|H_k\|_{2,\gamma}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt \\ &= \frac{1}{4} \|f\|_{2,\gamma}^2. \end{aligned}$$

Therefore, using Proposition 1.1,

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\lambda-1/2,\lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt \\ &= \lim_{\lambda \rightarrow \infty} \frac{1}{4} \|f_\lambda\|_{2,\lambda}^2 = \frac{1}{4} \|f\|_{2,\gamma}^2 = \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt. \end{aligned}$$

Now, taking $\alpha = \beta = \lambda - 1/2$,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \|g^{(\lambda-1/2,\lambda-1/2)} f_\lambda\|_{2,\lambda}^2 &= \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2,\lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt \\ &\quad + \lim_{\lambda \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\lambda-1/2,\lambda-1/2)} f_\lambda(x) \right)^2 d\mu_\lambda(x) \right) dt \\ &= \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{\partial}{\partial t} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt \\ &\quad + \int_0^\infty t \left(\int_{-\infty}^\infty \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial x} P_t^\gamma f(x) \right)^2 d\gamma(x) \right) dt = \|g^\gamma f\|_{2,\gamma}^2. \end{aligned}$$

now, by the L^2 continuity of $g^{(\lambda-1/2,\lambda-1/2)}$, we have

$$\|g^{(\lambda-1/2,\lambda-1/2)} f_\lambda\|_{2,\lambda} \leq C_2 \|f_\lambda\|_{2,\lambda}$$

Finally,

$$\|g^\gamma f\|_{2,\gamma} = \lim_{\lambda \rightarrow \infty} \|g^{(\lambda-1/2, \lambda-1/2)} f_\lambda\|_{2,\lambda} \leq C_2 \lim_{\lambda \rightarrow \infty} \|f_\lambda\|_{2,\lambda} = C_2 \|f\|_{2,\gamma}. \quad \blacksquare$$

ii) Let $f \in L^2((0, \infty), \mu_\alpha)$, define $f_\beta(x) = f\left(\frac{\beta}{2}(1-x)\right)$, then analogously as in the Hermite case

$$\|g^{(\alpha,\beta)} f_\beta\|_{2,(\alpha,\beta)}^2 = \int_0^\infty t \left(\int_{-1}^1 \left(\left(\frac{\partial}{\partial t} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 + (1-x^2) \left(\frac{\partial}{\partial x} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 \right) d\mu_{\alpha,\beta}(x) \right) dt,$$

where,

$$\int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 d\mu_{\alpha,\beta}(x) \right) dt = \frac{1}{4} \int_{-1}^1 |f_\beta(x)|^2 d\mu_{\alpha,\beta}(x) = \frac{1}{4} \|f_\beta\|_{2,(\alpha,\beta)}^2$$

and

$$\int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 d\mu_{\alpha,\beta}(x) \right) dt = \frac{1}{4} \int_{-1}^1 |f_\beta(x)|^2 d\mu_{\alpha,\beta}(x) = \frac{1}{4} \|f_\beta\|_{2,(\alpha,\beta)}^2,$$

Let us study the first term. Using Proposition 1.1 and Lebesgue's dominated convergence theorem, interchanging the integral with the series, we get

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 d\mu_{\alpha,\beta}(x) \right) dt &= \lim_{\beta \rightarrow \infty} \frac{1}{4} \|f_\beta\|_{2,(\alpha,\beta)}^2 = \frac{1}{4} \|f\|_{2,\alpha}^2 \\ &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt = \int_0^\infty t \left(\int_0^\infty \left(\frac{\partial}{\partial t} P_t^\alpha f(x) \right)^2 d\mu_\alpha(x) \right) dt. \end{aligned}$$

Let us look now the second term. First of all observe that,

$$\begin{aligned} \|\sqrt{x} L_{k-1}^{\alpha+1}\|_{2,\alpha}^2 &= \int_0^\infty |\sqrt{x} L_{k-1}^{\alpha+1}(x)|^2 \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} dx = (\alpha+1) \int_0^\infty |L_{k-1}^{\alpha+1}(x)|^2 \frac{x^{\alpha+1} e^{-x}}{\Gamma(\alpha+2)} dx \\ &= (\alpha+1) \|L_{k-1}^{\alpha+1}\|_{2,(\alpha+1)}^2 = (\alpha+1) \frac{\Gamma(k-1+\alpha+1+1)}{\Gamma(\alpha+2)(k-1)!} = k \|L_k^\alpha\|_{2,\alpha}^2, \end{aligned}$$

then we get, using Parseval's identity

$$\begin{aligned} \int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 d\mu_\alpha(x) &= \sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 e^{-2tk^{1/2}} \|\sqrt{x} L_{k-1}^{\alpha+1}\|_{2,\alpha}^2 \\ &= \sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 k e^{-2tk^{1/2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha,\beta)} f_\beta(x) \right)^2 d\mu_{\alpha,\beta}(x) \right) dt &= \lim_{\beta \rightarrow \infty} \frac{1}{4} \|f_\beta\|_{2,(\alpha,\beta)}^2 = \frac{1}{4} \|f\|_{2,\alpha}^2 \\ &= \int_0^\infty t \left(\sum_{k=1}^\infty \left| \left\langle f, \frac{L_k^\alpha}{\|L_k^\alpha\|_{2,\alpha}} \right\rangle \right|^2 k e^{-2tk^{1/2}} \right) dt = \int_0^\infty t \left(\int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 d\mu_\alpha(x) \right) dt. \end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\beta \rightarrow \infty} \|g^{(\alpha, \beta)} f_\beta\|_{2, (\alpha, \beta)}^2 &= \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\frac{\partial}{\partial t} P_t^{(\alpha, \beta)} f_\beta(x) \right)^2 d\mu_{\alpha, \beta}(x) \right) dt \\
&\quad + \lim_{\beta \rightarrow \infty} \int_0^\infty t \left(\int_{-1}^1 \left(\sqrt{1-x^2} \frac{\partial}{\partial x} P_t^{(\alpha, \beta)} f_\beta(x) \right)^2 d\mu_{\alpha, \beta}(x) \right) dt \\
&= \int_0^\infty t \left(\int_0^\infty \left(\frac{\partial}{\partial t} P_t^\alpha f(x) \right)^2 d\mu_\alpha(x) \right) dt \\
&\quad + \int_0^\infty t \left(\int_0^\infty \left(\sqrt{x} \frac{\partial}{\partial x} P_t^\alpha f(x) \right)^2 d\mu_\alpha(x) \right) dt = \|g^\alpha f\|_{2, \alpha}^2
\end{aligned}$$

and now, by the L^2 continuity of $g^{(\alpha, \beta)}$, we have

$$\|g^{(\alpha, \beta)} f_\beta\|_{2, (\alpha, \beta)} \leq C_2 \|f_\beta\|_{2, (\alpha, \beta)}$$

Thus finally,

$$\|g^\alpha f\|_{2, \alpha} = \lim_{\beta \rightarrow \infty} \|g^{(\alpha, \beta)} f_\beta\|_{2, (\alpha, \beta)} \leq C_2 \lim_{\beta \rightarrow \infty} \|f_\beta\|_{2, (\alpha, \beta)} = C_2 \|f\|_{2, \alpha}. \quad \blacksquare$$

Now we are going to consider the general case $p \neq 2$. For the proof we will follow the argument given by Betancour et al in [2].

Theorem 2.2. *Let $\alpha, \beta > -1$ and $1 < p < \infty$, then the $L^p(\mu_{\alpha, \beta})$ boundedness for the Jacobi-Littlewood-Paley g function*

$$(2.4) \quad \|g^{(\alpha, \beta)} f\|_{p, (\alpha, \beta)} \leq C_p \|f\|_{p, (\alpha, \beta)}$$

implies

i) *The $L^p(\gamma)$ -boundedness for the Gaussian-Littlewood-Paley g function*

$$(2.5) \quad \|g^\gamma f\|_{p, \gamma} \leq C_p \|f\|_{p, \gamma}.$$

and

ii) *The $L^p(\mu_\alpha)$ -boundedness for the Laguerre-Littlewood-Paley g function*

$$(2.6) \quad \|g^\alpha f\|_{p, \alpha} \leq C_p \|f\|_{p, \alpha}.$$

Proof

i) Assume that the operator $g^{(\lambda-1/2, \lambda-1/2)}$ is bounded in $L^p([-1, 1], \mu_\lambda)$. Let $\phi \in C_0^\infty(\mathbb{R})$, for each $\lambda > 0$ define the function

$$\phi_\lambda(x) = \phi(\sqrt{\lambda}x),$$

$x \in \mathbb{R}$. Let λ big enough such that $\text{supp } \phi_\lambda$ is contained in $[-1, 1]$. In what follows λ will be taken satisfying that condition.

Now, from the boundedness of $g^{(\lambda-1/2, \lambda-1/2)}$ we have

$$\|g^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda\|_{p, \lambda} \leq C \|\phi_\lambda\|_{p, \lambda},$$

that is to say,

$$\left\| \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda|^2 dt \right]^{1/2} \right\|_{p, \lambda} \leq C \|\phi_\lambda\|_{p, \lambda}$$

where $\nabla_\lambda = \nabla_{(\lambda-1/2, \lambda-1/2)}$. Now making the change of variables $x = \frac{y}{\sqrt{\lambda}}$ and taking $Z(\lambda) = \frac{\lambda^{1/2}[\Gamma(\lambda)]^2 2^{2\lambda}}{2\pi\Gamma(2\lambda)}$ we have

$$\left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left| \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda(\frac{y}{\sqrt{\lambda}})|^2 dt \right]^{1/2} \right|^p Z(\lambda) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} dy \right\}^{1/p} \leq C \|\phi_\lambda\|_{p,\lambda},$$

which implies

$$\left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left| \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda(\frac{y}{\sqrt{\lambda}})|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/p-1/2p} e^{\frac{y^2}{2}} \right|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \leq C(Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda}.$$

On the other hand, analogously we have from the case $p = 2$,

$$\left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left| \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda(\frac{y}{\sqrt{\lambda}})|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2-1/4} e^{\frac{y^2}{2}} \right|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \leq C(Z(\lambda))^{-1/2} \|\phi_\lambda\|_{2,\lambda},$$

Now, define for any $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$, the functions

$$F_{\lambda,K}(y) = \begin{cases} \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda(\frac{y}{\sqrt{\lambda}})|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2-1/4} e^{\frac{y^2}{2}} & \text{if } |y| \leq K \\ 0 & \text{if } |y| > K \end{cases}$$

and

$$f_{\lambda,K}(y) = \begin{cases} \left[\int_0^\infty t |\nabla_\lambda P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda(\frac{y}{\sqrt{\lambda}})|^2 dt \right]^{1/2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/p-1/2p} e^{\frac{y^2}{2}} & \text{if } |y| \leq K \\ 0 & \text{if } |y| > K. \end{cases}$$

Observe that $F_{\lambda,K} = f_{\lambda,K} \Omega_\lambda$, where

$$\Omega_\lambda(y) = e^{\frac{y^2}{2} - \frac{y^2}{p}} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2 - \lambda/p - 1/4 + 1/2p},$$

for all $K \in \mathbb{N}$ and $\sqrt{\lambda} > K$. Moreover,

$$|\Omega_\lambda(y)| \leq e^{\frac{y^2}{2} - \frac{y^2}{p}} e^{-\frac{y^2}{\lambda}(\lambda/2 - \lambda/p - 1/4 + 1/2p)} = e^{y^2(\frac{1}{4\lambda} - \frac{1}{2p\lambda})}.$$

Therefore, if $p \leq 2$ $|\Omega_\lambda(y)| \leq 1$ and for $p > 2$

$$|\Omega_\lambda(y)| \leq e^{K^2(\frac{1}{4\lambda} - \frac{1}{2p\lambda})},$$

hence Ω_λ is bounded in $[-K, K]$. Now

$$(Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} = (Z(\lambda))^{-1/p} \left\{ \int_{-1}^1 |\phi_\lambda(x)|^p \frac{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}}{2\pi\Gamma(2\lambda)} (1-x^2)^{\lambda-1/2} dx \right\}^{1/p}.$$

Thus, making the change of variables $x = \frac{y}{\sqrt{\lambda}}$ we get,

$$\begin{aligned} (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} &= (Z(\lambda))^{-1/p} \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi_\lambda(\frac{y}{\sqrt{\lambda}})|^p Z(\lambda) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} dy \right\}^{1/p} \\ &= \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} dy \right\}^{1/p} \leq C \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} &\leq \lim_{\lambda \rightarrow \infty} C \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \\ &= C \left\{ \int_{-\infty}^{\infty} |\phi(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} = C \|\phi\|_{p,\gamma}, \end{aligned}$$

and moreover,

$$(2.7) \quad (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda} \leq C \|\phi\|_{p,\gamma}$$

On the other hand,

$$\begin{aligned} \|F_{\lambda,K}\|_{2,\gamma} &= \left\{ \int_{\mathbb{R}} |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} = \left\{ \int_{-K}^K |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \\ (2.8) \quad &\leq \left\{ \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} |F_{\lambda,K}(y)|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/2} \leq C (Z(\lambda))^{-1/2} \|\phi_\lambda\|_{2,\lambda}. \end{aligned}$$

Then, from (2.7) and (2.8), we have

$$\|F_{\lambda,K}\|_{2,\gamma} \leq C \|\phi\|_{2,\gamma}.$$

Analogously,

$$\|F_{\lambda,K}\|_{p,\gamma} = \left\{ \int_{\mathbb{R}} |F_{\lambda,K}(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} = \left\{ \int_{-K}^K |f_{\lambda,K} \Omega_\lambda(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p}.$$

Since Ω_λ is bounded in $[-k, k]$, we get

$$\|F_{\lambda,K}\|_{p,\gamma} \leq C \left\{ \int_{-\infty}^{\infty} |f_{\lambda,K}(y)|^p \frac{e^{-y^2}}{\sqrt{\pi}} dy \right\}^{1/p} \leq C (Z(\lambda))^{-1/p} \|\phi_\lambda\|_{p,\lambda}$$

Then, from (2.7) and (2.9) we have

$$\|F_{\lambda,K}\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma}$$

for all $\sqrt{\lambda} > K$. Thus, $F_{\lambda,K}$ is a bounded sequence in $L^2(\mathbb{R}, \gamma)$ and $L^p(\mathbb{R}, \gamma)$. By Bourbaki-Alaoglu's theorem, there exists a subsequence $(\lambda_j)_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} \lambda_j = \infty$ and functions $F_K \in L^2(\mathbb{R}, \gamma)$ and $f_K \in L^p(\mathbb{R}, \gamma)$ satisfying

- $F_{\lambda_j,K} \rightarrow F_K$, as $j \rightarrow \infty$, in the weak topology on $L^2(\mathbb{R}, \gamma)$
- $F_{\lambda_j,K} \rightarrow f_K$, as $j \rightarrow \infty$, in the weak topology on $L^p(\mathbb{R}, \gamma)$

Moreover, $\text{sop} F_K \cup \text{sop} f_K \subseteq [-K, K]$, and

$$(2.9) \quad \|F_K\|_{2,\gamma} \leq \lim_{j \rightarrow \infty} \|F_{\lambda_j,K}\|_{2,\gamma} \leq C \|\phi\|_{2,\gamma}.$$

Analogously one gets,

$$(2.10) \quad \|f_K\|_{p,\gamma} \leq C \|\phi\|_{p,\gamma}$$

Observe that, defining for every $k \in \mathbb{N}$,

$$\tau_K(g) = \int_{-\infty}^{\infty} g(x) \chi_{[-K,K]}(x) dx,$$

then, by Cauchy-Schwartz inequality $\tau_K \in (L^2(\mathbb{R}, \gamma))^*$, and therefore,

$$\begin{aligned} \int_{-K}^K F_K(x) dx &= \int_{-\infty}^{\infty} F_K(x) \chi_{[-K, K]}(x) dx = \tau_K(F_K) = \lim_{j \rightarrow \infty} \tau_K(F_{\lambda_j, K}) \\ &= \lim_{j \rightarrow \infty} \int_{-\infty}^{\infty} F_{\lambda_j, K}(x) \chi_{[-K, K]}(x) dx = \int_{-K}^K f_K(x) dx; \end{aligned}$$

thus, $F_K = f_K$ a.e. on $[-K, K]$, for all $K \in \mathbb{N}$, so $F_K = f_K$ a.e. on \mathbb{R} . Then from (2.10), we get

$$(2.11) \quad \|F_K\|_{p, \gamma} \leq C \|\phi\|_{p, \gamma},$$

and therefore, from (2.9) and (2.11), there exists an increasing sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset (0, \infty)$, with $\lim_{j \rightarrow \infty} \lambda_j = \infty$, and a function $F \in L^p(\mathbb{R}, \gamma) \cap L^2(\mathbb{R}, \gamma)$, such that

- For each $K \in \mathbb{N}$, $F_{\lambda_j, K} \rightarrow F$, as $j \rightarrow \infty$, in the weak topology of $L^2(\mathbb{R}, \gamma)$ and in the weak topology of $L^p(\mathbb{R}, \gamma)$, and
- $\|F\|_{p, \gamma} \leq C \|\phi\|_{p, \gamma}$.

On the other hand,

$$\begin{aligned} F_{\lambda, K}(y) &= \chi_{[-K, K]}(y) \left(\int_0^{\infty} t |\nabla_{\lambda} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}})|^2 (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} dt \right)^{1/2} \\ &= \chi_{[-K, K]}(y) \left[\int_0^{\infty} t \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) \right)^2 dt (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} \right. \\ &\quad \left. + \int_0^{\infty} t (1 - \frac{y^2}{\lambda}) \left(\frac{\partial}{\partial x} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) \right)^2 dt (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} \right]^{1/2}. \end{aligned}$$

Now, define

$$g_{1, \lambda} \phi(y) = \chi_{[-K, K]}(y) \left(\int_0^{\infty} t \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) \right)^2 dt \right) (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2}$$

and

$$g_{2, \lambda} \phi(y) = \chi_{[-K, K]}(y) \left(\int_0^{\infty} t (1 - \frac{y^2}{\lambda}) \left(\frac{\partial}{\partial y} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) \right)^2 dt \right) (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2}.$$

Let us study first the function $g_{1, \lambda}$. For a function $\phi \in L^2(\mathbb{R}, \gamma)$,

$$\sqrt{t} \frac{\partial}{\partial t} P_t^{(\alpha, \beta)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) = - \sum_{k=1}^{\infty} \left\langle \phi_{\lambda}, \frac{P_k^{(\alpha, \beta)}}{\|P_k^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2} \right\rangle (t \lambda_k)^{1/2} e^{-t \lambda_k^{1/2}} P_k^{(\alpha, \beta)}(\frac{y}{\sqrt{\lambda}})$$

which converges absolutely. Then, taking the Cauchy product, we obtain,

$$\begin{aligned} \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{(\alpha, \beta)} \phi_{\lambda}(\frac{y}{\sqrt{\lambda}}) \right)^2 &= \\ \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_{\lambda}, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle &t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha, \beta)}(\frac{y}{\sqrt{\lambda}}) P_{k-n}^{(\alpha, \beta)}(\frac{y}{\sqrt{\lambda}}). \end{aligned}$$

Then, for the Gegenbauer case, $\alpha = \beta = \lambda - 1/2$, taking $\hat{\lambda}_k = k(k + 2\lambda)$

$$\begin{aligned}
& \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 = \\
& \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \times \left[\frac{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)\Gamma(2\lambda)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)\Gamma(\lambda+1/2)\Gamma(k-n+2\lambda)} \right]^2 \frac{2(n+\lambda)[\Gamma(\lambda+1/2)]^2 n! \Gamma(n+2\lambda)}{\Gamma(2\lambda+1)\Gamma(n+\lambda+1/2)} \\
& \times \frac{2(k-n+\lambda)[\Gamma(\lambda+1/2)]^2 (k-n)! \Gamma(k-n+2\lambda)}{\Gamma(2\lambda+1)\Gamma(k-n+\lambda+1/2)} \\
& = \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \times \frac{(n+\lambda)n!(k-n)!(k-n+\lambda)}{\lambda^2 (2\lambda)_n (2\lambda)_{k-n}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 = \\
& \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_{1,\lambda} \phi(y) &= \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} dt.
\end{aligned}$$

Let us take,

$$\begin{aligned}
H_{\lambda,K}^{N,1}(y) &= \chi_{[-K,K]}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) (1 - \frac{y^2}{\lambda})^{\lambda/2-1/4} e^{\frac{y^2}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
g_{1,\lambda} \phi(y) &= \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} dt \\
& + \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) (1 - \frac{y^2}{\lambda})^{\lambda-1/2} e^{y^2} dt
\end{aligned}$$

$$= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n}^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) \\ \times \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} + \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2-1/4} e^{\frac{y^2}{2}}.$$

Now, we want to prove that for $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$,

$$\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\theta}{e^{(N+1)}}.$$

Indeed, by Minkowski integral inequality we have

$$\left(\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \leq \int_0^\infty \left(\int_{-\infty}^\infty \left| H_{\lambda,K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} dt,$$

then,

$$\int_{-\infty}^\infty \left| H_{\lambda,K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ \leq \frac{1}{\sqrt{\pi}} \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \right. \\ \left. \times C_n^\lambda(x) C_{k-n}^\lambda(x) \right|^2 (1-x^2)^{\lambda-1/2} dx.$$

Using Gasper's linearization of the product of Jacobi polynomials, see [4],

$$C_n^\lambda(x) C_{k-n}^\lambda(x) = \frac{[\Gamma(\lambda + 1/2)]^2 \Gamma(n + 2\lambda) \Gamma(k - n + 2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n + \lambda + 1/2) \Gamma(k - n + \lambda + 1/2)} \\ \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) p_n^{(\lambda-1/2, \lambda-1/2)}(x) p_{k-n}^{(\lambda-1/2, \lambda-1/2)}(x) \\ = \frac{[\Gamma(\lambda + 1/2)]^2 \Gamma(n + 2\lambda) \Gamma(k - n + 2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n + \lambda + 1/2) \Gamma(k - n + \lambda + 1/2)} \\ \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \sum_{i=|k-2n|}^k \nu(i, n, k-n) p_i^{(\lambda-1/2, \lambda-1/2)}(x),$$

where, $p_i^{(\alpha, \beta)}(x) = P_i^{(\alpha, \beta)}(x) / P_i^{(\alpha, \beta)}(1)$ and

$$\nu(i, k-n, n) = \frac{1}{\|p_i^{(\alpha, \beta)}\|_{2, (\alpha, \beta)}^2} \int_{-1}^1 p_i^{(\alpha, \beta)}(x) p_{k-n}^{(\alpha, \beta)}(x) p_n^{(\alpha, \beta)}(x) d\mu_{(\alpha, \beta)}(x).$$

Hence,

$$\int_{-\infty}^\infty \left| H_{\lambda,K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ \leq \frac{1}{\sqrt{\pi}} \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=|k-2n|}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle t \hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2} e^{-t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \right. \\ \left. \times \frac{[\Gamma(\lambda + 1/2)]^2 \Gamma(n + 2\lambda) \Gamma(k - n + 2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n + \lambda + 1/2) \Gamma(k - n + \lambda + 1/2)} \right. \\ \left. \times P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \nu(i, n, k-n) p_i^{(\lambda-1/2, \lambda-1/2)}(x) \right|^2 (1-x^2)^{\lambda-1/2} dx.$$

Then, by Parseval's identity and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\
& \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda[\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \frac{\|\phi_{\lambda}\|^2}{\|C_{k-n}^{\lambda}\|^2 \|C_n^{\lambda}\|^2} \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \left| \frac{[\Gamma(\lambda + 1/2)]^2 \Gamma(n + 2\lambda) \Gamma(k - n + 2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n + \lambda + 1/2) \Gamma(k - n + \lambda + 1/2)} \right|^2 \\
& \times \left| P_n^{(\lambda-1/2, \lambda-1/2)}(1) P_{k-n}^{(\lambda-1/2, \lambda-1/2)}(1) \right|^2 |\nu(i, n, k - n)|^2 \left\| p_i^{(\lambda-1/2, \lambda-1/2)} \right\|^2
\end{aligned}$$

Now,

$$\begin{aligned}
\left| \nu(i, k - n, n) \left\| p_i^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)} \right| & \leq \frac{1}{\left\| p_i^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}} \int_{-1}^1 \left| p_i^{(\alpha, \beta)}(x) p_{k-n}^{(\alpha, \beta)}(x) \right| \left| p_n^{(\alpha, \beta)}(x) \right| d\mu_{(\alpha, \beta)}(x) \\
& \leq \frac{\binom{n+q}{n}}{\left\| p_i^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)} \left| P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1) \right|} \left\| p_i^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)} \left\| P_{k-n}^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)} \\
& = \frac{\binom{n+q}{n} \left\| P_{k-n}^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}}{\left| P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1) \right|},
\end{aligned}$$

where $q = \max(\alpha, \beta)$. Thus

$$\left| P_n^{(\alpha, \beta)}(1) P_{k-n}^{(\alpha, \beta)}(1) \right|^2 |\nu(i, k - n, n)|^2 \left\| p_i^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}^2 \leq \binom{n+q}{n}^2 \left\| P_{k-n}^{(\alpha, \beta)} \right\|_{2, (\alpha, \beta)}^2.$$

Therefore, for $\alpha = \beta = \lambda - 1/2$, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\
& \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda[\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \frac{\|\phi_{\lambda}\|^2}{\|C_{k-n}^{\lambda}\|^2 \|C_n^{\lambda}\|^2} \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \left| \frac{[\Gamma(\lambda + 1/2)]^2 \Gamma(n + 2\lambda) \Gamma(k - n + 2\lambda)}{[\Gamma(2\lambda)]^2 \Gamma(n + \lambda + 1/2) \Gamma(k - n + \lambda + 1/2)} \right|^2 \binom{n + \lambda - 1/2}{n}^2 \\
& \times \frac{[\Gamma(2\lambda)]^2 [\Gamma(n + \lambda + 1/2)]^2}{[\Gamma(\lambda + 1/2)]^2 [\Gamma(n + 2\lambda)]^2} \|C_{k-n}^{\lambda}\|^2 \\
& \times \frac{[\Gamma(\lambda + 1/2)]^2 [\Gamma(k - n + 2\lambda)]^2}{[\Gamma(2\lambda)]^2 [\Gamma(k - n + \lambda + 1/2)]^2} \frac{[\Gamma(n + \lambda + 1/2)]^2 \Gamma(2\lambda)(n + \lambda)n!}{[\Gamma(\lambda + 1/2)]^2 [n!]^2 \lambda \Gamma(n + 2\lambda)} \\
& = \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2\lambda)}{\lambda[\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \|\phi_{\lambda}\|^2 \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\
& \times \frac{[\Gamma(k - n + 2\lambda)]^2 [\Gamma(n + \lambda + 1/2)]^2 (n + \lambda)}{\Gamma(2\lambda) [\Gamma(k - n + \lambda + 1/2)]^2 n! \lambda \Gamma(n + 2\lambda)}.
\end{aligned}$$

On the other hand, using the Stirling's approximation formula for the gamma function,

$$\Gamma(az + b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-1/2} \quad (|arg z| < \pi, a > 0)$$

we obtain, for λ big enough that

$$\begin{aligned} & \frac{[\Gamma(k-n+2\lambda)]^2 [\Gamma(n+\lambda+1/2)]^2 (n+\lambda)}{\Gamma(2\lambda) [\Gamma(k-n+\lambda+1/2)]^2 n! \lambda \Gamma(n+2\lambda)} \\ & \sim \frac{2\pi e^{-4\lambda} (2\lambda)^{4\lambda+2k-2n-1} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2n} (n+\lambda)}{\sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda-1/2} 2\pi e^{-2\lambda} (\lambda)^{2\lambda+2k-2n} \sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda+n-1/2} \lambda n!} = \frac{2^{2k}}{2^{3n}} \frac{\lambda^{n-1} (n+\lambda)}{n!}. \end{aligned}$$

Then, for λ big enough

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ & \leq \frac{1}{\sqrt{\pi}} \frac{2\pi \Gamma(2\lambda) \|\phi_\lambda\|^2}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} \\ & \quad \times e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k}}{2^{3n}} \frac{\lambda^{n-1} (n+\lambda)}{n!}. \end{aligned}$$

Also,

$$t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k}}{2^{3n}} \frac{\lambda^{n-1} (n+\lambda)}{n!} \leq \frac{e^{-t} e^{-t\hat{\lambda}_{k-n}^{1/2}} 2^{2k} \lambda^n (\frac{n}{\lambda} + 1)}{t^2 2^{3n} n!}.$$

Taking $\lambda = (k-n)(1+2k)^2$, for $t \geq 1$ we get

$$\frac{1}{e^{t\hat{\lambda}_{k-n}^{1/2}}} \leq \frac{1}{e^{(k-n)^{1/2} (k-n+2(k-n)(1+2k)^2)^{1/2}}} \leq \frac{1}{e^{(k-n)(1+2(1+2k)^2)^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}.$$

Now, if $0 < t < 1$ we have that for t near to 0, exists $k > 0$ big enough such that $\frac{1}{k} \leq t$, i.e. $\frac{1}{t^2} \leq k^2$. Hence,

$$\frac{1}{e^{t\hat{\lambda}_{k-n}^{1/2}}} \leq \frac{1}{e^{\frac{1}{k}(k-n)^{1/2} (k-n+2(k-n)(1+2k)^2)^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}.$$

Therefore, for λ big enough there exists $C > 0$ such that

$$t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k}}{2^{3n}} \frac{\lambda^{n-1} (n+\lambda)}{n!} \leq \frac{e^{-tC}}{e^k}.$$

Then, for λ big enough we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N,1}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ & \leq \frac{1}{\sqrt{\pi}} \frac{2\pi \Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 \hat{\lambda}_n \hat{\lambda}_{k-n} \\ & \quad \times e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \frac{2^{2k}}{2^{3n}} \frac{\lambda^{n-1} (n+\lambda)}{n!} \\ & \leq \frac{e^{-t}}{\sqrt{\pi}} \frac{2\pi \Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \frac{MC \|\phi_\lambda\|^2}{e^{(N+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 \\ & \leq \frac{2\pi \Gamma(2\lambda)}{\lambda [\Gamma(\lambda)]^2 2^{2\lambda}} \frac{e^{-t} MC \|\phi_\lambda\|_{2,\lambda}^4}{e^{(N+1)}} \leq \frac{e^{-t} MC \|\phi\|_{2,\gamma}^4}{e^{(N+1)}}, \end{aligned}$$

Hence,

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \left| \int_0^{\infty} H_{\lambda,K}^{N,1}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} &\leq \int_0^{\infty} \left(\frac{e^{-t} MC \|\phi\|_{2,\gamma}^4}{e^{(N+1)t}} \right)^{1/2} dt \\ &= (MC)^{1/2} \frac{\|\phi\|_{2,\gamma}^2}{e^{\frac{(N+1)}{2}}} \int_0^{\infty} e^{-\frac{t}{2}} dt = \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{\frac{(N+1)}{2}}}. \end{aligned}$$

Thus, $\left\{ \int_0^{\infty} H_{\lambda,K}^{N,1} dt \right\}$ is a bounded sequence on $L^2(\mathbb{R}, \gamma)$, so by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\left\{ \int_0^{\infty} H_{\lambda_j,K}^{N,1} dt \right\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R}, \gamma)$ to a function $H_K^{N,1} \in L^2(\mathbb{R}, \gamma)$. Moreover,

$$(2.12) \quad \int_{-\infty}^{\infty} \left| H_K^{N,1}(y) \right|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \leq \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{\frac{(N+1)}{2}}}.$$

Then, there exists a non decreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that,

$$(2.13) \quad H_K^{N_j,1}(y) \longrightarrow 0, \quad a.e. \ y \in \mathbb{R}.$$

The study of $g_{2,\lambda}$ is essentially analogous, so fewer details will be given.

For $\phi \in L^2(\mathbb{R}, \gamma)$ we have

$$\frac{\partial}{\partial y} P_t^{(\alpha,\beta)} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) = - \sum_{k=1}^{\infty} \left\langle \phi_{\lambda}, \frac{P_k^{(\alpha,\beta)}}{\|P_k^{(\alpha,\beta)}\|_{2,(\alpha,\beta)}^2} \right\rangle e^{-t\lambda_k^{1/2}} \frac{(k + \alpha + \beta + 1)}{2} P_{k-1}^{(\alpha+1,\beta+1)} \left(\frac{y}{\sqrt{\lambda}} \right),$$

which converges absolutely. Then, again taking the Cauchy product, we obtain

$$\begin{aligned} &\left(\frac{\partial}{\partial y} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_{\lambda}, C_n^{\lambda} \rangle \langle \phi_{\lambda}, C_{k-n}^{\lambda} \rangle e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \frac{(n+2\lambda)(k-n+2\lambda)}{4} \\ &\quad \times \frac{\Gamma(2\lambda+2)\Gamma(n+\lambda+1/2)}{\Gamma(\lambda+3/2)\Gamma(n+2\lambda+1)} \frac{\Gamma(2\lambda+2)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+3/2)\Gamma(k-n+2\lambda+1)} \frac{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)} \\ &\quad \times \frac{\Gamma(2\lambda)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(k-n+2\lambda)} \frac{2(n+\lambda)[\Gamma(\lambda+1/2)]^2 n! \Gamma(n+2\lambda)}{\Gamma(2\lambda+1)[\Gamma(n+\lambda+1/2)]^2} \\ &\quad \times \frac{2(k-n+\lambda)[\Gamma(\lambda+1/2)]^2 (k-n)! \Gamma(k-n+2\lambda)}{\Gamma(2\lambda+1)[\Gamma(k-n+\lambda+1/2)]^2} \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^k \langle \phi_{\lambda}, C_n^{\lambda} \rangle \langle \phi_{\lambda}, C_{k-n}^{\lambda} \rangle e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \\ &\quad \times \frac{4(n+\lambda)n!(k-n+\lambda)(k-n)!}{(2\lambda)_n (2\lambda)_{k-n}}, \end{aligned}$$

then

$$\begin{aligned} &t \left(1 - \frac{y^2}{\lambda} \right) \left(\frac{\partial}{\partial t} P_t^{(\lambda-1/2, \lambda-1/2)} \phi_{\lambda} \left(\frac{y}{\sqrt{\lambda}} \right) \right)^2 \\ &= \left(1 - \frac{y^2}{\lambda} \right) \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right). \end{aligned}$$

Hence,

$$g_{2,\lambda}\phi(y) = \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} dt.$$

Taking,

$$H_{\lambda,K}^{N,2}(y) = \chi_{[-K,K]}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/4} e^{\frac{y^2}{2}}$$

we can write,

$$g_{2,\lambda}\phi(y) = \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} dt \\ + \chi_{[-K,K]}(y) \int_0^\infty \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} dt (1 - \frac{y^2}{\lambda})^{\lambda+1/2} e^{y^2} dt \\ = \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \\ \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} \\ + \int_0^\infty H_{\lambda,K}^{N,2}(y) dt (1 - \frac{y^2}{\lambda})^{\lambda+1/4} e^{\frac{y^2}{2}}.$$

Similarly the previous case, we want to prove that for any $K \in \mathbb{N}$ and $\lambda > 0$ such that $\sqrt{\lambda} > K$,

$$\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\theta}{e^{(N+1)}}.$$

Indeed, by Minkowski integral inequality we have

$$\left(\int_{-\infty}^\infty \left| \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} \leq \int_0^\infty \left(\int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} dt.$$

Now,

$$\int_{-\infty}^\infty |H_{\lambda,K}^{N,2}(y)|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ \leq \frac{1}{\sqrt{\pi}} \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \right. \\ \left. \times C_{n-1}^{\lambda+1}(x) C_{k-n-1}^{\lambda+1}(x) \right|^2 (1-x^2)^{\lambda-1/2} dx$$

Again, using the Gasper's linearization of the product of Jacobi polynomials,[4], we may write

$$\begin{aligned}
& C_{n-1}^{\lambda+1}(x)C_{k-n-1}^{\lambda+1}(x) \\
&= \frac{[\Gamma(\lambda+3/2)]^2\Gamma(n+2\lambda+1)\Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2\Gamma(n+2\lambda+3/2)\Gamma(k-n+2\lambda+3/2)} \\
&\quad \times P_{n-1}^{(\lambda+1/2,\lambda+1/2)}(1)P_{k-n-1}^{(\lambda+1/2,\lambda+1/2)}(1)p_{n-1}^{(\lambda+1/2,\lambda+1/2)}(x)p_{k-n-1}^{(\lambda+1/2,\lambda+1/2)}(x) \\
&= \frac{[\Gamma(\lambda+3/2)]^2\Gamma(n+2\lambda+1)\Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2\Gamma(n+2\lambda+3/2)\Gamma(k-n+2\lambda+3/2)} \\
&\quad \times P_{n-1}^{(\lambda+1/2,\lambda+1/2)}(1)P_{k-n-1}^{(\lambda+1/2,\lambda+1/2)}(1) \sum_{i=|k-2n|}^{k-2} \nu(i, n-1, k-n-1) p_i^{(\lambda+1/2,\lambda+1/2)}(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda,K}^{N,2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\
& \leq \frac{1}{\sqrt{\pi}} \int_{-1}^1 \left| \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^{k-2} \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|^2} \right\rangle \left\langle \phi_{\lambda}, \frac{C_{k-n}^{\lambda}}{\|C_{k-n}^{\lambda}\|^2} \right\rangle 4\lambda^2 t e^{-t(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \right. \\
& \quad \times \frac{[\Gamma(\lambda+3/2)]^2\Gamma(n+2\lambda+1)\Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2\Gamma(n+2\lambda+3/2)\Gamma(k-n+2\lambda+3/2)} P_{n-1}^{(\lambda+1/2,\lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2,\lambda+1/2)}(1) \\
& \quad \left. \times \nu(i, n-1, k-n-1) p_i^{(\lambda+1/2,\lambda+1/2)}(x) \right|^2 (1-x^2)^{\lambda-1/2} dx.
\end{aligned}$$

Then, using Parseval's identity and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda,K}^{N,2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\
& \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \frac{\|\phi_{\lambda}\|^2}{\|C_n^{\lambda}\|^2 \|C_{k-n}^{\lambda}\|^2} \\
& \quad \times 16\lambda^4 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left| \frac{[\Gamma(\lambda+3/2)]^2\Gamma(n+2\lambda+1)\Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2\Gamma(n+2\lambda+3/2)\Gamma(k-n+2\lambda+3/2)} \right|^2 \\
& \quad \times \left| P_{n-1}^{(\lambda+1/2,\lambda+1/2)}(1) P_{k-n-1}^{(\lambda+1/2,\lambda+1/2)}(1) \right|^2 \left| \nu(i, n-1, k-n-1) \right|^2 \left\| p_i^{(\lambda+1/2,\lambda+1/2)} \right\|^2.
\end{aligned}$$

Then, by by a similar argument as in the previous case, we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda,K}^{N,2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\
& \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_{\lambda}, \frac{C_n^{\lambda}}{\|C_n^{\lambda}\|} \right\rangle \right|^2 \frac{\|\phi_{\lambda}\|^2}{\|C_n^{\lambda}\|^2 \|C_{k-n}^{\lambda}\|^2} \\
& \quad \times 16\lambda^4 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left| \frac{[\Gamma(\lambda+3/2)]^2\Gamma(n+2\lambda+1)\Gamma(k-n+2\lambda+1)}{[\Gamma(2\lambda+2)]^2\Gamma(n+2\lambda+3/2)\Gamma(k-n+2\lambda+3/2)} \right|^2 \\
& \quad \times \left(\frac{n+\lambda-1/2}{n-1} \right)^2 \left[\frac{\Gamma(2\lambda)\Gamma(k-n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(k-n+2\lambda)} \right]^2 \frac{(2\lambda+2)(2\lambda+1)(k-n)}{(\lambda+1/2)^2(k-n+2\lambda)} \|C_{k-n}^{\lambda}\|_{\lambda}^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned} & \left| \frac{[\Gamma(\lambda + 3/2)]^2 \Gamma(n + 2\lambda + 1) \Gamma(k - n + 2\lambda + 1)}{[\Gamma(2\lambda + 2)]^2 \Gamma(n + 2\lambda + 3/2) \Gamma(k - n + 2\lambda + 3/2)} \right|^2 \\ & \times \binom{n + \lambda - 1/2}{n - 1}^2 \left[\frac{\Gamma(2\lambda) \Gamma(k - n + \lambda + 1/2)}{\Gamma(\lambda + 1/2) \Gamma(k - n + 2\lambda)} \right]^2 \frac{(2\lambda + 2)(2\lambda + 1)(k - n)}{(\lambda + 1/2)^2 (k - n + 2\lambda)} \|C_{k-n}^\lambda\|_\lambda^2 \\ & = \frac{2[\Gamma(n + 2\lambda)]^2 [\Gamma(n + \lambda + 1/2)]^2 [\Gamma(k - n + \lambda + 1/2)]^2 \|C_{k-n}^\lambda\|_\lambda^2}{(2\lambda)^4 [\Gamma(2\lambda)]^2 [\Gamma(n + 2\lambda + 1/2)]^2 [\Gamma(k - n + 2\lambda + 1/2)]^2 [\Gamma(n)]^2}. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N, 2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \\ & \leq \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda + 1))}{(\lambda + 1)[\Gamma(\lambda + 1)]^2 2^{2(\lambda + 1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 \frac{\|\phi_\lambda\|^2 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})}}{\|C_n^\lambda\|^2} \\ & \times \frac{2[\Gamma(n + 2\lambda)]^2 [\Gamma(n + \lambda + 1/2)]^2 [\Gamma(k - n + \lambda + 1/2)]^2 \|C_{k-n}^\lambda\|_\lambda^2}{[\Gamma(2\lambda)]^2 [\Gamma(n + 2\lambda + 1/2)]^2 [\Gamma(k - n + 2\lambda + 1/2)]^2 [\Gamma(n)]^2} \\ & = \frac{1}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda + 1)) \|\phi_\lambda\|^2}{(\lambda + 1)[\Gamma(\lambda + 1)]^2 2^{2(\lambda + 1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \\ & \times \frac{2[\Gamma(n + 2\lambda)] [\Gamma(n + \lambda + 1/2)]^2 [\Gamma(k - n + \lambda + 1/2)]^2 (n + \lambda)n!}{\Gamma(2\lambda) [\Gamma(n + 2\lambda + 1/2)]^2 [\Gamma(k - n + 2\lambda + 1/2)]^2 [\Gamma(n)]^2 \lambda}. \end{aligned}$$

Then, for λ big enough we have

$$\begin{aligned} & \frac{2[\Gamma(n + 2\lambda)] [\Gamma(n + \lambda + 1/2)]^2 [\Gamma(k - n + \lambda + 1/2)]^2 (n + \lambda)n!}{\Gamma(2\lambda) [\Gamma(n + 2\lambda + 1/2)]^2 [\Gamma(k - n + 2\lambda + 1/2)]^2 [\Gamma(n)]^2 \lambda} \\ & \sim \frac{2\sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda + n - 1/2} 2\pi e^{-2\lambda} (\lambda)^{2\lambda + 2n} 2\pi e^{-2\lambda} (\lambda)^{2\lambda + 2k - 2n} (n + \lambda)n!}{\sqrt{2\pi} e^{-2\lambda} (2\lambda)^{2\lambda - 1/2} 2\pi e^{-4\lambda} (2\lambda)^{4\lambda + 2n} 2\pi e^{-4\lambda} (2\lambda)^{4\lambda + 2k - 2n} \lambda [(n - 1)!]^2} \\ & = \frac{2^{n+1} \lambda^{n-1} (n + \lambda)n!}{2^{2k+8\lambda} \lambda^{4\lambda} e^{-4\lambda} [(n - 1)!]^2} \leq \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n - 1)!}. \end{aligned}$$

Therefore, for λ big enough we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| H_{\lambda, K}^{N, 2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\|\phi_\lambda\|^2}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda + 1))}{(\lambda + 1)[\Gamma(\lambda + 1)]^2 2^{2(\lambda + 1)}} \\ & \times \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n - 1)!}. \end{aligned}$$

Similarly the previous case, taking $\lambda = (k - n)(1 + 2k)^2$ we have for k big enough there exists $C > 0$ such that

$$t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda} (n - 1)!} \leq \frac{e^{-tC}}{e^k}.$$

Thus, for λ big enough we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| H_{\lambda,K}^{N,2}(y) \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \leq \frac{\|\phi_\lambda\|^2}{\sqrt{\pi}} \frac{2\pi\Gamma(2(\lambda+1))}{(\lambda+1)[\Gamma(\lambda+1)]^2 2^{2(\lambda+1)}} \\
& \times \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 t^2 e^{-2t(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})} \left(\frac{n}{\lambda} + 1 \right) \frac{\lambda^n n}{\lambda^{4\lambda}(n-1)!} \\
& \leq \frac{e^{-t}}{\sqrt{\pi}} \frac{2\pi(2\lambda+1)2\lambda\Gamma(2\lambda)}{(\lambda+1)\lambda^2[\Gamma(\lambda)]^2 2^{2\lambda} 2^2} \frac{MC\|\phi_\lambda\|^2}{e^{(N+1)}} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=|k-2n|}^k \left| \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|} \right\rangle \right|^2 \\
& \leq \frac{2\pi\Gamma(2\lambda)}{\lambda[\Gamma(\lambda)]^2 2^{2\lambda}} \frac{e^{-t} MC\|\phi_\lambda\|_{2,\lambda}^4}{e^{(N+1)}} \leq \frac{e^{-t} MC\|\phi\|_{2,\gamma}^4}{e^{(N+1)}},
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} \left| \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \right|^2 e^{-y^2} \frac{dy}{\sqrt{\pi}} \right)^{1/2} & \leq \int_0^\infty \left(\frac{e^{-t} MC\|\phi\|_{2,\gamma}^4}{e^{(N+1)}} \right)^{1/2} dt \\
& = \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{\frac{(N+1)}{2}}}.
\end{aligned}$$

Then, $\left\{ \int_0^\infty H_{\lambda,K}^{N,2} dt \right\}$ is a bounded sequence on $L^2(\mathbb{R}, \gamma)$, so by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\left\{ \int_0^\infty H_{\lambda_j,K}^{N,2} dt \right\}_{j \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R}, \gamma)$ to a function $H_K^{N,2} \in L^2(\mathbb{R}, \gamma)$. Moreover,

$$(2.14) \quad \int_{-\infty}^{\infty} \left| H_K^{N,2}(y) \right|^2 \frac{e^{-y^2}}{\sqrt{\pi}} dy \leq \frac{(MC)^{1/2} \|\phi\|_{2,\gamma}^2}{e^{\frac{(N+1)}{2}}}$$

Therefore, there exists a non decreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that,

$$(2.15) \quad H_K^{N_j,2}(y) \longrightarrow 0, \quad a.e. \ y \in \mathbb{R}.$$

On the other hand,

$$\begin{aligned}
(F_{\lambda,K}(y))^2 & = g_{1,\lambda}\phi(y) + g_{2,\lambda}\phi(y) \\
& = \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}} \right) \\
& \quad \times \left(1 - \frac{y^2}{\lambda} \right)^{\lambda-1/2} e^{y^2} + \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{\frac{y^2}{2}} \\
& \quad + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \\
& \quad \times C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}} \right) \left(1 - \frac{y^2}{\lambda} \right)^{\lambda+1/2} e^{y^2} \\
& \quad + \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{\frac{y^2}{2}} \\
& = F_{\lambda,K}^N(y) + \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2-1/4} e^{\frac{y^2}{2}} \\
& \quad + \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \left(1 - \frac{y^2}{\lambda} \right)^{\lambda/2+1/4} e^{\frac{y^2}{2}}
\end{aligned}$$

where,

$$\begin{aligned}
F_{\lambda,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n}^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) \\
&\quad \times \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \\
&\quad \times C_{n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2}
\end{aligned}$$

Defining, for each $m \in \mathbb{N}$, $F_K^{N_m} = F^2 - H_K^{N_m,1} - H_K^{N_m,2}$, then since $F_{\lambda,K} \rightarrow F_K$, as $j \rightarrow \infty$ in the weak topology of $L^2(\mathbb{R}, \gamma)$ and that

$$F_{\lambda,K}^N = (F_{\lambda,K})^2 - \int_0^\infty H_{\lambda,K}^{N,1}(y) dt \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2-1/4} e^{\frac{y^2}{2}} - \int_0^\infty H_{\lambda,K}^{N,2}(y) dt \left(1 - \frac{y^2}{\lambda}\right)^{\lambda/2+1/4} e^{\frac{y^2}{2}},$$

we have, for all $m \in \mathbb{N}$,

$$F_{\lambda_j,K}^{N_m} \longrightarrow F_K^{N_m}, \quad \text{weakly in } L^2(\mathbb{R}, \gamma), \quad \text{as } j \rightarrow \infty$$

also,

$$(2.17) \quad F_K^{N_m}(y) \longrightarrow F^2(y), \quad \text{as } m \rightarrow \infty \quad a.e. \quad y \in \mathbb{R}$$

Now,

$$\begin{aligned}
F_{\lambda,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} C_n^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n}^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) \\
&\quad \times \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\lambda, \frac{C_n^\lambda}{\|C_n^\lambda\|^2} \right\rangle \left\langle \phi_\lambda, \frac{C_{k-n}^\lambda}{\|C_{k-n}^\lambda\|^2} \right\rangle \frac{4\lambda^2}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \\
&\quad \times C_{n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2} \\
&= \sum_{k=1}^N \sum_{n=1}^k \frac{(n+\lambda)n!(k-n)!(k-n+\lambda)}{\lambda^2(2\lambda)_n(2\lambda)_{k-n}} \frac{\hat{\lambda}_n^{1/2} \hat{\lambda}_{k-n}^{1/2}}{(\hat{\lambda}_n^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle \\
&\quad \times C_n^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n}^\lambda\left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} \\
&+ \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, C_n^\lambda \rangle \langle \phi_\lambda, C_{k-n}^\lambda \rangle C_{n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) C_{k-n-1}^{\lambda+1}\left(\frac{y}{\sqrt{\lambda}}\right) \\
&\quad \times \frac{4(n+\lambda)n!(k-n+\lambda)(k-n)!}{(2\lambda)_n(2\lambda)_{k-n}} \frac{1}{(\hat{\lambda}_k^{1/2} + \hat{\lambda}_{k-n}^{1/2})^2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^N \sum_{n=1}^k \frac{(\frac{n}{\lambda} + 1)n!(k-n)!(\frac{k-n}{\lambda} + 1)}{2(2 + \frac{1}{\lambda}) \dots (2 + \frac{n-1}{\lambda})2(2 + \frac{1}{\lambda}) \dots (2 + \frac{k-n-1}{\lambda})} \\
&\quad \times \frac{(n(\frac{n}{\lambda} + 2))^{1/2}((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}}{\{(n(\frac{n}{\lambda} + 2))^{1/2} + ((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}\}^2} \langle \phi_\lambda, \lambda^{-\frac{n}{2}} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \rangle \\
&\quad \times \lambda^{-\frac{n}{2}} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}}\right) \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} \\
&+ \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, \lambda^{-\frac{n}{2}} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \rangle \lambda^{-\frac{n-1}{2}} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) \lambda^{-\frac{k-n-1}{2}} C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) \\
&\quad \times \frac{4(\frac{n}{\lambda} + 1)n!(\frac{k-n}{\lambda} + 1)(k-n)!}{2(2 + \frac{1}{\lambda}) \dots (2 + \frac{n-1}{\lambda})2(2 + \frac{1}{\lambda}) \dots (2 + \frac{k-n-1}{\lambda})} \\
&\quad \times \frac{1}{\{(n(\frac{n}{\lambda} + 2))^{1/2} + ((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}\}^2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\lim_{\lambda \rightarrow \infty} F_{\lambda, K}^N(y) \\
&= \lim_{\lambda \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \frac{(\frac{n}{\lambda} + 1)n!(k-n)!(\frac{k-n}{\lambda} + 1)}{2(2 + \frac{1}{\lambda}) \dots (2 + \frac{n-1}{\lambda})2(2 + \frac{1}{\lambda}) \dots (2 + \frac{k-n-1}{\lambda})} \\
&\quad \times \frac{(n(\frac{n}{\lambda} + 2))^{1/2}((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}}{\{(n(\frac{n}{\lambda} + 2))^{1/2} + ((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}\}^2} \langle \phi_\lambda, \lambda^{-\frac{n}{2}} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \rangle \\
&\quad \times \lambda^{-\frac{n}{2}} C_n^\lambda \left(\frac{y}{\sqrt{\lambda}}\right) \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \left(\frac{y}{\sqrt{\lambda}}\right) \left(1 - \frac{y^2}{\lambda}\right)^{\lambda-1/2} e^{y^2} \\
&+ \lim_{\lambda \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \langle \phi_\lambda, \lambda^{-\frac{n}{2}} C_n^\lambda \rangle \langle \phi_\lambda, \lambda^{-\frac{k-n}{2}} C_{k-n}^\lambda \rangle \lambda^{-\frac{n-1}{2}} C_{n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) \lambda^{-\frac{k-n-1}{2}} C_{k-n-1}^{\lambda+1} \left(\frac{y}{\sqrt{\lambda}}\right) \\
&\quad \times \frac{4(\frac{n}{\lambda} + 1)n!(\frac{k-n}{\lambda} + 1)(k-n)!}{2(2 + \frac{1}{\lambda}) \dots (2 + \frac{n-1}{\lambda})2(2 + \frac{1}{\lambda}) \dots (2 + \frac{k-n-1}{\lambda})} \\
&\quad \times \frac{1}{\{(n(\frac{n}{\lambda} + 2))^{1/2} + ((k-n)(\frac{k-n}{\lambda} + 2))^{1/2}\}^2} \left(1 - \frac{y^2}{\lambda}\right)^{\lambda+1/2} e^{y^2} \\
&= \sum_{k=1}^N \sum_{n=1}^k \frac{n!(k-n)!}{2^n 2^{k-n}} \frac{n^{1/2}(k-n)^{1/2}}{(n^{1/2} + (k-n)^{1/2})^2} \left\langle \phi, \frac{H_n}{n!} \right\rangle \left\langle \phi, \frac{H_{k-n}}{(k-n)!} \right\rangle \frac{H_n(y)}{n!} \frac{H_{k-n}(y)}{(k-n)!} \\
&+ \sum_{k=1}^N \sum_{n=1}^k \frac{n!(k-n)!}{2^n 2^{k-n}} \frac{4}{2(n^{1/2} + (k-n)^{1/2})^2} \\
&\quad \times \left\langle \phi, \frac{H_n}{n!} \right\rangle \left\langle \phi, \frac{H_{k-n}}{(k-n)!} \right\rangle \frac{H_{n-1}(y)}{(n-1)!} \frac{H_{k-n-1}(y)}{(k-n-1)!} \\
&= \sum_{k=1}^N \sum_{n=1}^k n^{1/2}(k-n)^{1/2} \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_n(y) H_{k-n}(y) \\
&\quad \times \int_0^\infty t e^{-t(n^{1/2} + (k-n)^{1/2})} dt
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^N \sum_{n=1}^k \frac{4n(k-n)}{2} \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_{n-1}(y) H_{k-n-1}(y) \\
& \times \int_0^\infty t e^{-t(n^{1/2} + (k-n)^{1/2})} dt \\
& = \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k n^{1/2} (k-n)^{1/2} t e^{-t(n^{1/2} + (k-n)^{1/2})} \\
& \quad \times \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_n(y) H_{k-n}(y) dt \\
& + \int_0^\infty \sum_{k=1}^N \sum_{n=1}^k \frac{4n(k-n)}{2} t e^{-t(n^{1/2} + (k-n)^{1/2})} \\
& \quad \times \left\langle \phi, \frac{H_n}{\|H_n\|^2} \right\rangle \left\langle \phi, \frac{H_{k-n}}{\|H_{k-n}\|^2} \right\rangle H_{n-1}(y) H_{k-n-1}(y) dt \\
& = \int_0^\infty \left(\sqrt{t} \sum_{k=1}^N k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt \\
& + \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^N 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \right)^2 dt,
\end{aligned}$$

so we have,

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} F_{\lambda, K}^N(y) &= \int_0^\infty \left(\sqrt{t} \sum_{k=1}^N k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt \\
&+ \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^N 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \right)^2 dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
F_K^{N_m}(y) &= \int_0^\infty \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt \\
&+ \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^{N_m} 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \right)^2 dt.
\end{aligned}$$

On the other hand, given that

$$|H_k(x)| \leq \eta e^{\frac{x^2}{2}} 2^{\frac{k}{2}} \sqrt{k!}, \quad \eta = 1.086435$$

we get,

$$\begin{aligned}
\left| k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right| &\leq k^{1/2} e^{-tk^{1/2}} \|\phi\| \frac{\eta e^{\frac{y^2}{2}} 2^{\frac{k}{2}} \sqrt{k!}}{\pi^{1/4} 2^{\frac{k}{2}} \sqrt{k!}} \\
&= \frac{\eta \|\phi\| e^{\frac{y^2}{2}}}{\pi^{1/4}} k^{1/2} e^{-tk^{1/2}},
\end{aligned}$$

then,

$$\begin{aligned} \left| t \left(\sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 \right| &\leq \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \right)^2 \\ &\leq \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2. \end{aligned}$$

Now, for $t \geq 1$ we have

$$\sum_{k=1}^{\infty} k^{1/2} e^{-t(k^{1/2}-1/2)} \leq \sum_{k=1}^{\infty} k^{1/2} e^{-k^{1/2}+1/2},$$

which converges. Therefore

$$\begin{aligned} \int_1^{\infty} t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 dt &\leq \int_1^{\infty} t e^{-t} \left(\sum_{k=1}^{\infty} k^{1/2} e^{-t(k^{1/2}-1/2)} \right)^2 dt \\ &\leq \int_1^{\infty} t e^{-t} \left(\sum_{k=1}^{\infty} k^{1/2} e^{-k^{1/2}+1/2} \right)^2 dt \leq C \int_1^{\infty} t e^{-t} dt < \infty. \end{aligned}$$

Let $0 < t < 1$. For $y \in [-K, K]$, let us define

$$\tau_m(y) = \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y).$$

We know that for $y \in [-K, K]$,

$$\tau_m(y) \longrightarrow \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y), \quad \text{if } m \rightarrow \infty,$$

and also,

$$\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) = \frac{\partial}{\partial t} P_t^\gamma(\phi(y)).$$

Now, for $\varepsilon = 1$, exists $M(\varepsilon, y) > 0$ such that

$$(2.18) \quad |\tau_m(y) - \frac{\partial}{\partial t} P_t^\gamma(\phi(y))| < 1, \quad \text{if } m \geq M,$$

then, for $m \geq M$ we get

$$|\tau_m(y)| < 1 + \left| \frac{\partial}{\partial t} P_t^\gamma(\phi(y)) \right| < t(1 + C(1 + |y|)e^{-t}).$$

Therefore, for $y \in [-K, K]$ we have

$$\int_0^1 t(1 + C(1 + |y|)e^{-t})^2 dt < \infty.$$

Let us then define

$$\psi(t) = \begin{cases} \frac{\eta^2 \|\phi\|^2 e^{y^2}}{\pi^{1/2}} t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 & \text{if } t \geq 1 \\ t(1 + C(1 + |y|)e^{-t})^2 & \text{if } 0 < t < 1 \end{cases}$$

which is integrable in $(0, \infty)$, now for all $m > 0$ we have

$$\left| \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 \right| \leq \psi(t), \quad \text{for all } t > 0,$$

then, by Lebesgue's dominated convergence theorem we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^\infty \left(\sqrt{t} \sum_{k=1}^{N_m} k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt \\ &= \int_0^\infty \left(\sqrt{t} \sum_{k=1}^\infty k^{1/2} e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_k(y) \right)^2 dt = \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^\gamma(\phi(y)) \right)^2 dt. \end{aligned}$$

Similarly, we have

$$\lim_{m \rightarrow \infty} \int_0^\infty t \left(\frac{1}{\sqrt{2}} \sum_{k=1}^{N_m} 2k e^{-tk^{1/2}} \left\langle \phi, \frac{H_k}{\|H_k\|^2} \right\rangle H_{k-1}(y) \right)^2 dt = \int_0^\infty t \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial y} P_t^\gamma(\phi(y)) \right)^2 dt$$

i.e.,

$$(2.19) \quad F_K^{N_m}(y) \rightarrow (g^\gamma \phi(y))^2, \quad \text{cuando } m \rightarrow \infty \text{ a.e. } y \in \mathbb{R}$$

therefore, from (2.17) and (2.19) we get

$$F(y) = g^\gamma(\phi(y))$$

ii) The proof is essentially analogous to i), so fewer details will be provided.

Assume that the operator $g^{(\alpha, \beta)}$ is bounded in $L^p([-1, 1], \mu_{\alpha, \beta})$. Let $\phi \in C_0^\infty(0, \infty)$ and $\beta > 0$, then we have

$$\|g^{(\alpha, \beta)} \phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})} \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})}$$

where, $\phi_\beta(x) = \phi\left(\frac{\beta}{2}(1-x)\right)$, for $x \in [-1, 1]$, i.e.,

$$\left\| \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta|^2 dt \right\}^{1/2} \right\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})} \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_{(\alpha, \beta)})}$$

Now making the change of variable $x = 1 - \frac{2}{\beta}y$, we have

$$\begin{aligned} & \left\{ \eta_{\alpha, \beta} \frac{2^{\alpha+\beta+1}}{\beta^{\alpha+1}} \int_0^\beta \left| \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta(1 - \frac{2}{\beta}y)|^2 dt \right\}^{1/2} (1 - \frac{y}{\beta})^{\beta/p} e^{\frac{y}{p}} \right|^p y^\alpha e^{-y} dy \right\}^{1/p} \\ & \leq C \|\phi_\beta\|_{L^p([-1, 1], \mu_\lambda)} \end{aligned}$$

thus,

$$\begin{aligned} & \left\{ \int_0^\beta \left| \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta(1 - \frac{2}{\beta}y)|^2 dt \right\}^{1/2} (1 - \frac{y}{\beta})^{\beta/p} e^{\frac{y}{p}} \right|^p y^\alpha e^{-y} dy \right\}^{1/p} \\ & \leq C (Z_{\alpha, \beta})^{-1} \|\phi_\beta\|_{L^p([-1, 1], \mu_\lambda)} \end{aligned}$$

where, $Z_{\alpha, \beta} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\beta^{\alpha+2}\Gamma(\beta)}$. Analogously we get,

$$\begin{aligned} & \left\{ \int_0^\beta \left| \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta(1 - \frac{2}{\beta}y)|^2 dt \right\}^{1/2} (1 - \frac{y}{\beta})^{\beta/2} e^{\frac{y}{2}} \right|^2 y^\alpha e^{-y} dy \right\}^{1/2} \\ & \leq C (Z_{\alpha, \beta})^{-1} \|\phi_\beta\|_{L^2([-1, 1], \mu_\lambda)} \end{aligned}$$

Now, for each $K \in \mathbb{N}$ and $\beta > 0$, such that $\beta > K$, define the functions

$$F_{\beta, K}(y) = \chi_{(0, K)}(y) \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta(1 - \frac{2}{\beta}y)|^2 dt \right\}^{1/2} (1 - \frac{y}{\beta})^{\beta/2} e^{\frac{y}{2}}$$

and

$$f_{\beta, K}(y) = \chi_{(0, K)}(y) \left\{ \int_0^\infty t |\nabla_{(\alpha, \beta)} P_t^{(\alpha, \beta)} \phi_\beta(1 - \frac{2}{\beta}y)|^2 dt \right\}^{1/2} (1 - \frac{y}{\beta})^{\beta/p} e^{\frac{y}{p}}$$

From the previous inequalities both series converge for all $y \in (0, \beta)$, and $F_{\beta,K} = f_{\beta,K} \Omega_{\beta}$, where

$$\Omega_{\beta}(y) = e^{\frac{y}{2} - \frac{y}{p}} \left(1 - \frac{y}{\beta}\right)^{\beta/2 - \beta/p}$$

for all $K \in \mathbb{N}$ and $\beta > K$. Now as,

$$|\Omega_{\beta}(y)| = e^{\frac{y}{2} - \frac{y}{p}} \left(1 - \frac{y}{\beta}\right)^{\beta/2 - \beta/p} \leq e^{\frac{y}{2} - \frac{y}{p}} (e^{-y/\beta})^{\beta/2 - \beta/p} = 1$$

we conclude that Ω_{β} is bounded in $(0, K)$. On the other hand,

$$(Z_{\alpha,\beta})^{-1/p} \|\phi_{\beta}\|_{L^p([-1,1], \mu_{(\alpha,\beta)})} = (Z_{\alpha,\beta})^{-1/p} \left\{ \eta_{\alpha,\beta} \int_{-1}^1 |\phi_{\beta}(x)|^p (1-x)^{\alpha} (1+x)^{\beta} dx \right\}^{1/p},$$

and making the change of variable $x = 1 - \frac{2}{\beta}y$ we get

$$\begin{aligned} & (Z_{\alpha,\beta})^{-1/p} \left\{ \eta_{\alpha,\beta} \int_{-1}^1 |\phi_{\beta}(x)|^p (1-x)^{\alpha} (1+x)^{\beta} dx \right\}^{1/p} \\ &= (Z_{\alpha,\beta})^{-1/p} \left\{ \eta_{\alpha,\beta} \frac{2}{\beta} \int_0^{\beta} \left| \phi_{\beta}\left(1 - \frac{2}{\beta}y\right) \right|^p \left(\frac{2y}{\beta}\right)^{\alpha} \left(2 - \frac{2y}{\beta}\right)^{\beta} dy \right\}^{1/p} \\ &\leq C \left\{ \int_0^{\beta} |\phi(y)|^p y^{\alpha} \left(1 - \frac{y}{\beta}\right)^{\beta} dy \right\}^{1/p} \leq C \left\{ \int_0^{\infty} |\phi(y)|^p y^{\alpha} e^{-y} dy \right\}^{1/p} = C \|\phi\|_{L^p((0,\infty), \mu_{\alpha})}. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} (Z_{\alpha,\beta})^{-1/p} \|\phi_{\beta}\|_{L^p([-1,1], \mu_{(\alpha,\beta)})} &\leq \lim_{\beta \rightarrow \infty} C \left\{ \int_0^{\infty} |\phi(y)|^p y^{\alpha} e^{-y} dy \right\}^{1/p} \\ &= C \|\phi\|_{L^p((0,\infty), \mu_{\alpha})}. \end{aligned}$$

Moreover,

$$(2.20) \quad (Z_{\alpha,\beta})^{-1/p} \|\phi_{\beta}\|_{L^p([-1,1], \mu_{(\alpha,\beta)})} \leq C \|\phi\|_{L^p((0,\infty), \mu_{\alpha})}.$$

Now,

$$\|F_{\beta,K}\|_{L^2((0,\infty), \mu_{\alpha})} \leq C (Z_{\alpha,\beta})^{-1/2} \|\phi_{\beta}\|_{L^2([-1,1], \mu_{(\alpha,\beta)})}$$

and therefore, from (2.20) for $p = 2$ and (2.21), we get

$$\|F_{\beta,K}\|_{L^2((0,\infty), \mu_{\alpha})} \leq C \|\phi\|_{L^2((0,\infty), \mu_{\alpha})}.$$

Analogously, using that Ω_{β} is bounded in $(0, K)$,

$$\begin{aligned} \|F_{\beta,K}\|_{L^p((0,\infty), \mu_{\alpha})} &\leq C \left\{ \frac{1}{\Gamma(\alpha+1)} \int_0^k |f_{\beta,K}(y)|^p y^{\alpha} e^{-y} dy \right\}^{1/p} \\ (2.21) \quad &\leq C (Z_{\alpha,\beta})^{-1/p} \|\phi_{\beta}\|_{L^p([-1,1], \mu_{(\alpha,\beta)})}. \end{aligned}$$

Then, from (2.20) and (2.21),

$$\|F_{\beta,K}\|_{L^p((0,\infty), \mu_{\alpha})} \leq C \|\phi\|_{L^p((0,\infty), \mu_{\alpha})},$$

for all $\beta > K$. Therefore $\{F_{\beta,K}\}$ is a bounded subsequence in $L^2((0,\infty), \mu_{\alpha})$ with $L^p((0,\infty), \mu_{\alpha})$. Thus by the Bourbaki-Alaoglu's theorem, there exists an increasing sequence $\{\beta_j\}_{j \in \mathbb{N}}$ with $\lim_{j \rightarrow \infty} \beta_j = \infty$ and functions $F_K \in L^2((0,\infty), \mu_{\alpha})$ and $f_K \in L^p((0,\infty), \mu_{\alpha})$ satisfying that

- $F_{\beta_j,K} \rightarrow F_K$, as $j \rightarrow \infty$, in the weak topology of $L^2((0,\infty), \mu_{\alpha})$
- $F_{\beta_j,K} \rightarrow f_K$, as $j \rightarrow \infty$, in the weak topology of $L^p((0,\infty), \mu_{\alpha})$.

Then, as in i), we can conclude that there exists an increasing sequence $(\beta_j)_{j \in \mathbb{N}} \subset (0, \infty)$ such that $\lim_{j \rightarrow \infty} \beta_j = \infty$, and a function $F \in L^p((0, \infty), \mu_\alpha) \cap L^2((0, \infty), \mu_\alpha)$, such that

- for each $K \in \mathbb{N}$, $F_{\beta_j, K} \rightarrow F$, as $j \rightarrow \infty$, in the weak topology on $L^2((0, \infty), \mu_\alpha)$ and in the weak topology on $L^p((0, \infty), \mu_\alpha)$.
- $\|F\|_{L^p((0, \infty), \mu_\alpha)} \leq C \|\phi\|_{L^p((0, \infty), \mu_\alpha)}$.

Analogously to the Hermite case, we have

$$\begin{aligned} F_{\beta, K}(y) &= \chi_{(0, K)}(y) \left\{ \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 dt \left(1 - \frac{y}{\beta}\right)^\beta e^y \right. \\ &\quad \left. + \int_0^\infty \frac{t}{\beta} \left(\frac{\partial}{\partial y} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 dt 4y \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y \right\}^{1/2}. \end{aligned}$$

Now, let us define

$$g_{1, \beta} \phi(y) = \chi_{(0, K)}(y) \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 dt \left(1 - \frac{y}{\beta}\right)^\beta e^y$$

and

$$g_{2, \beta} \phi(y) = \chi_{(0, K)}(y) \int_0^\infty \frac{t}{\beta} \left(\frac{\partial}{\partial y} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 dt 4y \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y$$

Let us first $g_{1, \beta}$.

For a function $\phi \in L^2((0, \infty), \mu_\alpha)$

$$\begin{aligned} \left(\sqrt{t} \frac{\partial}{\partial t} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 &= \sum_{k=1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} \\ &\quad \times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right). \end{aligned}$$

Therefore, as in the previous case

$$\begin{aligned} g_{1, \beta} \phi(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \\ &\quad \times P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^\beta e^y \\ &\quad + \int_0^\infty H_{\beta, K}^{N, 1}(y) dt \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2}, \end{aligned}$$

where,

$$\begin{aligned} H_{\beta, K}^{N, 1}(y) &= \chi_{(0, K)}(y) \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} \\ &\quad \times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^{\beta/2} e^{y/2}. \end{aligned}$$

We want to prove that for $K \in \mathbb{N}$ and $\beta > K$,

$$\int_0^\infty \left| \int_0^\infty H_{\beta, K}^{N, 1}(y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha + 1)} \leq \frac{C \|\phi\|_{2, \alpha}^4}{e^{N+1}}.$$

Now,

$$\begin{aligned}
& \int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
& \leq \frac{\beta^\alpha}{2^{\alpha+\beta}\Gamma(\alpha+1)} \int_0^\infty \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \right. \\
& \quad \left. \times P_n^{(\alpha,\beta)}(1 - \frac{2}{\beta}y) P_{k-n}^{(\alpha,\beta)}(1 - \frac{2}{\beta}y) \right|^2 \left(\frac{2y}{\beta} \right)^\alpha 2^\beta (1 - \frac{y}{\beta})^\beta dy \\
& = \frac{\beta^{\alpha+1}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \int_{-1}^1 \left| \sum_{k=N+1}^\infty \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle t \lambda_n^{1/2} \lambda_{k-n}^{1/2} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \right. \\
& \quad \left. \times P_n^{(\alpha,\beta)}(x) P_{k-n}^{(\alpha,\beta)}(x) \right|^2 (1-x)^\alpha (1+x)^\beta dx.
\end{aligned}$$

Then, again by Gasper's linearization of the product of Jacobi polynomials [4] and using Parseval's identity, we have

$$\begin{aligned}
& \int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
& \leq \frac{\beta^{\alpha+1}}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \sum_{i=k-2n}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \left| \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 t^2 \lambda_n \lambda_{k-n} \\
& \quad \times \left| P_n^{(\alpha,\beta)}(1) P_{k-n}^{(\alpha,\beta)}(1) \right|^2 e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} |\nu(i, k-n, n)|^2 \|p_i^{(\alpha,\beta)}\|_2^2.
\end{aligned}$$

Using analogous boundedness argument as in the Hermite case, we get

$$\begin{aligned}
& \int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\
& \leq \frac{\beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \frac{t^2 \lambda_n \lambda_{k-n} 2n \binom{n+q}{n}^2 e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})}}{\|P_n^{(\alpha,\beta)}\|_2^2} \\
& \leq \frac{e^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \right|^2 \frac{2n \binom{n+q}{n}^2 e^{-t\lambda_{k-n}^{1/2}}}{t^2 \|P_n^{(\alpha,\beta)}\|_2^2}.
\end{aligned}$$

Now, for β big enough

$$\frac{1}{\|P_n^{(\alpha,\beta)}\|_2^2} = \frac{(2n + \alpha + \beta + 1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \leq \frac{Cn!}{(\alpha+1)_n}$$

and

$$(2.22) \quad \binom{n+q}{n} = \binom{n+\beta}{n} = \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)n!} \sim \frac{\beta^n}{n!}.$$

Thus, for β big enough,

$$\begin{aligned} & \int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ & \leq \frac{C e^{-t} \beta^{\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2n \beta^{2n} e^{-t \lambda_{k-n}^{1/2}}}{t^2 (\alpha+1)_n} \end{aligned}$$

Taking $\beta = (k-n)(1+2k^2)^2$, then, k is big enough if β is big enough and we have two cases.

- Case $t \geq 1$: In this case, we have

$$\frac{1}{e^{t \lambda_{k-n}^{1/2}}} \leq \frac{1}{e^{(k-n)^{1/2} (k-n + (k-n)(1+2k^2)^2 + \alpha + 1)^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k}.$$

Therefore, for k big enough exist $C > 0$ such that

$$\sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2n \beta^{2n} e^{-t \lambda_{k-n}^{1/2}}}{t^2 (\alpha+1)_n} \leq \frac{C}{e^N} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2,$$

thus,

$$(2.23) \quad \int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C e^{-t} \|\phi\|_{2,\alpha}^4}{e^{N+1}}$$

- Case $0 < t < 1$: for t near to 0, exists $k > 0$ big enough such that $\frac{1}{k} \leq t$, i.e. $\frac{1}{t^2} \leq k^2$. Then, analogously to the Hermite case

$$\frac{1}{e^{t \lambda_{k-n}^{1/2}}} \leq \frac{1}{e^{(k-n)}} \frac{1}{e^k} \frac{1}{e^k}.$$

Hence,

$$\int_0^\infty \left| H_{\beta,K}^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C e^{-t} \|\phi\|_{2,\alpha}^4}{e^{N+1}}.$$

Therefore,

$$\begin{aligned} \left(\int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,1}(y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \right)^{1/2} & \leq \int_0^\infty \left(\frac{e^{-t} C \|\phi\|_{2,\alpha}^4}{e^{(N+1)}} \right)^{1/2} dt \\ & = \frac{(C)^{1/2} \|\phi\|_{2,\alpha}^2}{e^{\frac{(N+1)}{2}}} \end{aligned}$$

Thus, $\left\{ \int_0^\infty H_{\beta,K}^{N,1} dt \right\}$ is a bounded sequence on $L^2((0, \infty), \mu_\alpha)$ so by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\left\{ \int_0^\infty H_{\beta_j,K}^{N,1} dt \right\}_{j \in \mathbb{N}}$ converges weakly in $L^2((0, \infty), \mu_\alpha)$ to a function $H_K^{N,1} \in L^2((0, \infty), \mu_\alpha)$. Moreover,

$$(2.24) \quad \int_0^\infty \left| H_K^{N,1}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C}{e^{\frac{(N+1)}{2}}}$$

Then, there exists a non decreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that,

$$(2.25) \quad H_K^{N_j,1}(y) \longrightarrow 0, \quad a.e. \ y \in \mathbb{R}.$$

For the function $g_{2,\beta}$ we get similar estimates. Given $\phi \in L^2((0, \infty), \mu_\alpha)$,

$$\begin{aligned} \left(\frac{\sqrt{t}}{\sqrt{\beta}} \frac{\partial}{\partial y} P_t^{(\alpha, \beta)} \phi_\beta \left(1 - \frac{2}{\beta} y\right) \right)^2 &= \sum_{k=1}^{\infty} \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n \lambda_{k-n}}{4n(k-n)} \\ &\quad \times P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) \frac{t}{\beta} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})}. \end{aligned}$$

Thus,

$$\begin{aligned} g_{2,\beta} \phi(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) \\ &\quad \times \frac{y}{\beta n(k-n)} \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y + \int_0^\infty H_{\beta, K}^{N, 2}(y) dt y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2}, \end{aligned}$$

where,

$$\begin{aligned} H_{\beta, K}^{N, 2}(y) &= \chi_{(0, K)}(y) \sum_{k=N+1}^{\infty} \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{t}{\beta} e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \frac{\lambda_n \lambda_{k-n}}{n(k-n)} \\ &\quad \times P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) y^{1/2} \left(1 - \frac{y}{\beta}\right)^{(\beta+1)/2} e^{y/2}. \end{aligned}$$

Again, we want to prove that for $K \in \mathbb{N}$ and $\beta > K$,

$$\int_0^\infty \left| \int_0^\infty H_{\beta, K}^{N, 2}(y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{(C)^{1/2} \|\phi\|_{2, \alpha}^2}{e^{\frac{(N+1)}{2}}}.$$

Let us see this,

$$\begin{aligned} &\int_0^\infty \left| H_{\beta, K}^{N, 2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ &\leq \frac{\beta^{\alpha+2}}{2^{\alpha+\beta+3} \Gamma(\alpha+1)} \int_{-1}^1 \left| \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=k-2n}^{k-2} \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{t}{\beta} \right. \\ &\quad \times e^{-t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \frac{\lambda_n \lambda_{k-n}}{n(k-n)} P_{n-1}^{(\alpha+1, \beta+1)}(1) P_{k-n-1}^{(\alpha+1, \beta+1)}(1) \nu(i, k-n-1, n-1) p_i^{(\alpha+1, \beta+1)}(x) \left. \right|^2 \\ &\quad \times (1-x)^{\alpha+1} (1+x)^{\beta+1} dx. \end{aligned}$$

Thus, using Parseval's identity

$$\begin{aligned} &\int_0^\infty \left| H_{\beta, K}^{N, 2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ &\leq \frac{\beta^{\alpha+2}}{\eta_{\alpha+1, \beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \sum_{k=N+1}^{\infty} \sum_{n=1}^k \sum_{i=k-2n}^{k-2} \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \right|^2 \left| \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \right|^2 \left| \frac{t}{\beta} \right|^2 \\ &\quad \times e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \left| \frac{\lambda_n \lambda_{k-n}}{n(k-n)} P_{n-1}^{(\alpha+1, \beta+1)}(1) P_{k-n-1}^{(\alpha+1, \beta+1)}(1) \right|^2 \\ &\quad \times |\nu(i, k-n-1, n-1)|^2 \left\| p_i^{(\alpha+1, \beta+1)} \right\|_{(\alpha+1, \beta+1)}^2. \end{aligned}$$

Analogously the previous case,

$$\begin{aligned} & \int_0^\infty \left| H_{\beta,K}^{N,2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ & \leq \frac{\beta^{\alpha+2} \|\phi_\beta\|^2}{\eta_{\alpha+1,\beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{t^2 2(n-1)}{\beta^2 \|P_n^{(\alpha,\beta)}\|_2^2 \|P_{k-n}^{(\alpha,\beta)}\|_2^2} \\ & \quad \times e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \left| \frac{\lambda_n \lambda_{k-n}}{n(k-n)} \right|^2 \cdot \left(\frac{n-1+q}{n-1} \right)^2 \|P_{k-n-1}^{(\alpha+1,\beta+1)}\|_{(\alpha+1,\beta+1)}^2. \end{aligned}$$

Also, given that

$$\|P_{k-n-1}^{(\alpha+1,\beta+1)}\|_{(\alpha+1,\beta+1)}^2 = \frac{(\alpha+\beta+3)(\alpha+\beta+2)(k-n)}{(\alpha+1)(\beta+1)(k-n+\alpha+\beta+1)} \|P_{k-n}^{(\alpha,\beta)}\|_{(\alpha,\beta)}^2,$$

we get,

$$\begin{aligned} & \int_0^\infty \left| H_{\beta,K}^{N,2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ & \leq \frac{\beta^{\alpha+2} \|\phi_\beta\|^2}{\eta_{\alpha+1,\beta+1} 2^{\alpha+\beta+3} \Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{t^2 2e^{-2t(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})} \lambda_n^2 \lambda_{k-n}}{\beta^2 n} \\ & \quad \times \frac{\left(\frac{n-1+q}{n-1} \right)^2 (\alpha+\beta+3)(\alpha+\beta+2)}{(\alpha+1)(\beta+1) \|P_n^{(\alpha,\beta)}\|_2^2} \\ & \leq \frac{e^{-t\beta\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2e^{-t\lambda_{k-n}^{1/2}} (n+\alpha+\beta+1)}{t^2 \beta} \frac{\left(\frac{n-1+q}{n-1} \right)^2}{\|P_n^{(\alpha,\beta)}\|_2^2}. \end{aligned}$$

Then we have for $\beta > 0$ big enough

$$\frac{\left(\frac{n-1+q}{n-1} \right)^2 (n+\alpha+\beta+1)}{\beta \|P_n^{(\alpha,\beta)}\|_2^2} \leq \frac{Cn\beta^{2(n-1)}}{(\alpha+1)_n} \leq \frac{Cn\beta^{2n}}{(\alpha+1)_n}.$$

Hence, for $\beta > 0$ big enough

$$\begin{aligned} & \int_0^\infty \left| H_{\beta,K}^{N,2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \\ & \leq \frac{Ce^{-t\beta\alpha+1} \|\phi_\beta\|^2}{\eta_{\alpha,\beta} 2^{\alpha+\beta+1} \Gamma(\alpha+1)} \sum_{k=N+1}^\infty \sum_{n=1}^k \left| \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2} \right\rangle \right|^2 \frac{2n\beta^{2n} e^{-t\lambda_{k-n}^{1/2}}}{t^2 (\alpha+1)_n} \end{aligned}$$

then, analogous to (2.23) we have

$$\int_0^\infty \left| H_{\beta,K}^{N,2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{Ce^{-t} \|\phi\|_{2,\alpha}^4}{e^{N+1}},$$

therefore,

$$\left(\int_0^\infty \left| \int_0^\infty H_{\beta,K}^{N,2}(y) dt \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \right)^{1/2} \leq \frac{(C)^{1/2} \|\phi\|_{2,\alpha}^2}{e^{\frac{(N+1)}{2}}}.$$

Thus, $\left\{ \int_0^\infty H_{\beta,K}^{N,2} dt \right\}$ is a bounded sequence on $L^2((0, \infty), \mu_\alpha)$, so by Bourbaki-Alaoglu's theorem, there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ $\lim_{j \rightarrow \infty} \lambda_j = \infty$ such that, for all $N \in \mathbb{N}$, $\left\{ \int_0^\infty H_{\beta,K}^{N,2} dt \right\}_{j \in \mathbb{N}}$ converges weakly in $L^2((0, \infty), \mu_\alpha)$ to a function $H_K^{N,2} \in L^2((0, \infty), \mu_\alpha)$. Moreover,

$$(2.26) \quad \int_0^\infty \left| H_K^{N,2}(y) \right|^2 y^\alpha e^{-y} \frac{dy}{\Gamma(\alpha+1)} \leq \frac{C}{e^{\frac{(N+1)}{2}}}$$

Then, there exists a non decreasing sequence $(N_j)_{j \in \mathbb{N}}$ such that,

$$(2.27) \quad H_K^{N_j,2}(y) \longrightarrow 0, \quad a.e. \ y \in (0, \infty).$$

Therefore, similarly to (2.16)

$$\begin{aligned} (F_{\beta,K}(y))^2 &= g_{1,\beta} \phi(y) + g_{2,\beta} \phi(y) \\ &= F_{\beta,K}^N(y) + \int_0^\infty H_{\beta,K}^{N,1}(y) dt (1 - \frac{y}{\beta})^{\beta/2} e^{y/2} + \int_0^\infty H_{\beta,K}^{N,2}(y) dt y^{1/2} (1 - \frac{y}{\beta})^{(\beta+1)/2} e^{y/2}, \end{aligned}$$

where

$$\begin{aligned} F_{\beta,K}^N(y) &= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \\ &\quad \times P_n^{(\alpha,\beta)} (1 - \frac{2}{\beta} y) P_{k-n}^{(\alpha,\beta)} (1 - \frac{2}{\beta} y) (1 - \frac{y}{\beta})^\beta e^y \\ &\quad + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha,\beta)}}{\|P_n^{(\alpha,\beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha,\beta)}}{\|P_{k-n}^{(\alpha,\beta)}\|_2^2} \right\rangle \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \\ &\quad \times \frac{y}{\beta n(k-n)} P_{n-1}^{(\alpha+1,\beta+1)} (1 - \frac{2}{\beta} y) P_{k-n-1}^{(\alpha+1,\beta+1)} (1 - \frac{2}{\beta} y) (1 - \frac{y}{\beta})^{\beta+1} e^y. \end{aligned}$$

Defining, for each $m \in \mathbb{N}$, $F_K^{N_m} = F^2 - H_K^{N_m,1} - H_K^{N_m,2}$, then since $F_{\beta,K} \rightarrow F_K$, as $j \rightarrow \infty$ in the weak topology of $L^2((0, \infty), \mu_\alpha)$ and that

$$F_{\beta,K}^N = (F_{\beta,K})^2 - \int_0^\infty H_{\beta,K}^{N,1}(y) dt (1 - \frac{y}{\beta})^{\beta/2} e^{y/2} - \int_0^\infty H_{\beta,K}^{N,2}(y) dt y^{1/2} (1 - \frac{y}{\beta})^{(\beta+1)/2} e^{y/2},$$

we have, for all $m \in \mathbb{N}$,

$$F_{\beta_j,K}^{N_m} \longrightarrow F_K^{N_m}, \quad \text{weakly in } L^2((0, \infty), \mu_\alpha), \quad \text{as } j \rightarrow \infty,$$

and also,

$$(2.28) \quad F_K^{N_m}(y) \longrightarrow F^2(y), \quad \text{as } m \rightarrow \infty \quad a.e. \ y \in (0, \infty).$$

Now, given that

$$\lim_{\beta \rightarrow \infty} P_{n-1}^{(\alpha+1,\beta+1)} (1 - \frac{2}{\beta} y) = \lim_{\beta \rightarrow \infty} P_{n-1}^{(\alpha+1,\beta+1)} (1 - \frac{2y}{\beta+1} \frac{\beta+1}{\beta}) = L_{n-1}^{\alpha+1}(y)$$

and

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{1}{\|P_n^{(\alpha,\beta)}\|_{(\alpha,\beta)}^2} &= \lim_{\beta \rightarrow \infty} \left(\frac{2n}{\beta} + \frac{\alpha}{\beta} + 1 + \frac{1}{\beta} \right) n! \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \frac{\beta^{\alpha+1} \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\beta+n+\alpha+1)}{\beta^\alpha \Gamma(n+\beta+1)} \\ &= \frac{n! \Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} = \frac{1}{\|L_n^\alpha\|_\alpha^2}; \end{aligned}$$

we get,

$$\begin{aligned}
\lim_{\beta \rightarrow \infty} F_{\beta, K}^N(y) &= \lim_{\beta \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n^{1/2} \lambda_{k-n}^{1/2}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \\
&\quad \times P_n^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) P_{k-n}^{(\alpha, \beta)} \left(1 - \frac{2}{\beta} y\right) \left(1 - \frac{y}{\beta}\right)^\beta e^y \\
&+ \lim_{\beta \rightarrow \infty} \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi_\beta, \frac{P_n^{(\alpha, \beta)}}{\|P_n^{(\alpha, \beta)}\|_2^2} \right\rangle \left\langle \phi_\beta, \frac{P_{k-n}^{(\alpha, \beta)}}{\|P_{k-n}^{(\alpha, \beta)}\|_2^2} \right\rangle \frac{\lambda_n \lambda_{k-n}}{(\lambda_k^{1/2} + \lambda_{k-n}^{1/2})^2} \\
&\quad \times \frac{y}{\beta n(k-n)} P_{n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{2}{\beta} y\right) P_{k-n-1}^{(\alpha+1, \beta+1)} \left(1 - \frac{y}{\beta}\right)^{\beta+1} e^y \\
&= \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \left\langle \phi, \frac{L_{k-n}^\alpha}{\|L_{k-n}^\alpha\|_\alpha^2} \right\rangle \frac{n^{1/2} (k-n)^{1/2}}{(n^{1/2} + (k-n)^{1/2})^2} L_n^\alpha(y) L_{k-n}^\alpha(y) \\
&\quad + \sum_{k=1}^N \sum_{n=1}^k \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \left\langle \phi, \frac{L_{k-n}^\alpha}{\|L_{k-n}^\alpha\|_\alpha^2} \right\rangle \frac{y}{(n^{1/2} + (k-n)^{1/2})^2} L_{n-1}^{\alpha+1}(y) L_{k-n-1}^{\alpha+1}(y) \\
&= \int_0^\infty t \left(\sum_{k=1}^N \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_n^\alpha(y) \right)^2 dt \\
&\quad + \int_0^\infty t \left(\sum_{k=1}^N \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \sqrt{y} e^{-tk^{1/2}} L_{n-1}^{\alpha+1}(y) \right)^2 dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
F_K^{N_m}(y) &= \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(y) \right)^2 dt \\
&\quad + \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle \sqrt{y} e^{-tk^{1/2}} L_{k-1}^{\alpha+1}(y) \right)^2 dt.
\end{aligned}$$

We want to prove that

$$(2.29) \quad F_K^{N_m}(y) \rightarrow (g^\alpha \phi(y))^2, \quad \text{as } m \rightarrow \infty \quad a.e. \quad y \in \mathbb{R}$$

Indeed, initially we have

$$\left| \sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right| \leq \|\phi\| \sum_{k=1}^\infty k^{1/2} e^{-tk^{1/2}} \frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha}.$$

From (8.22.1) of [9], for α y $x > 0$, we have

$$L_k^\alpha(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \cos\{2(kx)^{1/2} - \alpha\pi/2 - \pi/4\} + O(k^{\alpha/2-3/4}).$$

Thus, there exist $M > 0$ such that

$$\left| L_k^\alpha(x) - \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} k^{\alpha/2-1/4} \cos\{2(kx)^{1/2} - \alpha\pi/2 - \pi/4\} \right| \leq M k^{\alpha/2-3/4},$$

hence,

$$|L_k^\alpha(x)| \leq k^{\alpha/2-1/4} \left(M + \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} \right).$$

On the other hand, sin for k big enough $\|L_k^\alpha\|_\alpha^2 \approx \frac{k^\alpha}{\Gamma(\alpha+1)}$, then, there exist $N_1 > 0$ such that

$$\frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha} \leq \frac{(M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}) [\Gamma(k+1)]^{1/2}}{k^{1/4}}, \quad \text{for any } k \geq N_1.$$

Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \frac{|L_k^\alpha(x)|}{\|L_k^\alpha\|_\alpha} &\leq (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}) [\Gamma(\alpha+1)]^{1/2} \sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \\ &\quad + (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}) [\Gamma(\alpha+1)]^{1/2} \sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}}. \end{aligned}$$

Then, for $t \geq 1$ we have

$$\begin{aligned} &\left| t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 \right| \\ &\leq (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 t \\ &\quad + (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left(\sum_{k=1}^{N_1} k^{1/4} e^{-k^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right) \\ &\quad + (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) t \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2. \end{aligned}$$

Now, analogously to the Hermite case, we have

$$\begin{aligned} &\int_1^\infty t \left\{ (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left(\sum_{k=1}^{N_1} k^{1/4} e^{-tk^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 \right. \\ &\quad + (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left(\sum_{k=1}^{N_1} k^{1/4} e^{-k^{1/2}} \frac{k^{\alpha/2}}{\|L_k^\alpha\|_\alpha} \right)^2 \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right) \\ &\quad \left. + (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left(\sum_{k=1}^{\infty} k^{1/2} e^{-tk^{1/2}} \right)^2 \right\} dt \\ &\leq (M + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4})^2 \Gamma(\alpha+1) \left\{ \left(\sum_{k=1}^{N_1} \frac{e^{-(k^{1/2}-1/2)} k^{\alpha/2+1/4}}{\|L_k^\alpha\|_\alpha} \right)^2 \right. \\ &\quad \left. + \left(\sum_{k=1}^{N_1} \frac{e^{-k^{1/2}} k^{\alpha/2+1/4}}{\|L_k^\alpha\|_\alpha} \right) \left(\sum_{k=1}^{\infty} k^{1/2} e^{-(k^{1/2}-1)} \right) + \left(\sum_{k=1}^{\infty} k^{1/2} e^{-(k^{1/2}-1/2)} \right)^2 \right\} \int_1^\infty t e^{-t} dt < \infty. \end{aligned}$$

For $0 < t < 1$, given that $|\frac{\partial}{\partial t} P_t^\alpha(\phi(y))| \leq C(1 + |y|)e^{-t}$ we get for $y \in (0, K)$

$$t \left| \sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right|^2 < t \left(1 + \left| \frac{\partial}{\partial t} P_t^\alpha(\phi(y)) \right| \right)^2 < t (1 + C(1 + |y|)e^{-t})^2,$$

where,

$$\int_0^1 t(1 + C(1 + |y|)e^{-t})^2 dt < \infty,$$

then, by Lebesgue's dominated convergence theorem we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 dt \\ &= \int_0^\infty t \left(\sum_{k=1}^\infty \left\langle \phi, \frac{L_k^\alpha}{\|L_k^\alpha\|_\alpha^2} \right\rangle k^{1/2} e^{-tk^{1/2}} L_k^\alpha(x) \right)^2 dt = \int_0^\infty t \left(\frac{\partial}{\partial t} P_t^\alpha(\phi(y)) \right)^2 dt. \end{aligned}$$

Similarly as before,

$$\lim_{m \rightarrow \infty} \int_0^\infty t \left(\sum_{k=1}^{N_m} \left\langle \phi, \frac{L_n^\alpha}{\|L_n^\alpha\|_\alpha^2} \right\rangle \sqrt{y} e^{-tk^{1/2}} L_{n-1}^{\alpha+1}(y) \right)^2 dt = \int_0^\infty t \left(\sqrt{y} \frac{\partial}{\partial y} P_t^\alpha(\phi(y)) \right)^2 dt.$$

Thus,

$$F_K^{N_m}(y) \rightarrow (g^\alpha \phi(y))^2, \text{ as } m \rightarrow \infty \text{ a.e. } y \in \mathbb{R},$$

hence, from (2.28) and (2.29) we have $F(y) = g^\alpha(\phi(y))$. ■

REFERENCES

- [1] Bayley, W. N. *Generalized hypergeometric series*. Cambridge Tracts in Mathematics and Mathematical Physics # 32. New York. (1964).
- [2] Betancor, J. Fariña, J. Rodriguez, L. Sanabria, A. *Transferring boundedness from conjugate operators associated with Jacobi, Laguerre and Fourier-Bessel expansions to conjugate operators in the Hankel setting*. J. Fourier Anal. Appl. 14 (2008), no. 4, 493–513.
- [3] Gasper, G. Unpublished manuscript.
- [4] Gasper, G. *Linearization of the product of Jacobi polynomials I*, Canad. J. Math. 22. (1970) 171–175
- [5] Gutiérrez, C. *On the Riesz transforms for the Gaussian measure*. J. Func. Anal. 120 (1) (1994) 107–134. MR1262249 (95c:35013)
- [6] Navas, E & Urbina, W. *A transference result of the L_p continuity of the Jacobi Riesz transform to the Gaussian and Laguerre Riesz transforms*. J. Fourier Anal. Appl. 19, Issue 5(2013), 910–942.
- [7] Nowak, A. *On Riesz transforms for Laguerre expansions*. J. Funct. Anal. 215 (2004), no. 1, 217–240.
- [8] A. Nowak & P. Sjögren *Riesz Transform for Jacobi Expansions*. J. Anal. Math. 104 (2008), 341–369
- [9] Szegő, G. *Orthogonal polynomials*. Colloq. Publ. 23. Amer. Math. Soc. Providence (1959).
- [10] Urbina, W. *Operators Semigroups associated to Classical Orthogonal Polynomials and Functional Inequalities*. Lecture Notes of the French Mathematical Society (SMF). 2008.

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